# Homework 1

(1.10) $(2n+1)+(2n+3)+(2n+5)+\dots+(4n-1)=3n^2$ $\quad n\in\mathbb{Z}$

$$\Rightarrow \sum_{k=1}^{n}(2n+2k-1)=3n^2.$$

Proof by induction:

$$P(n) = \text{"}\sum_{k=1}^{n}(2n+2k-1)=3n^2\text{"}$$

$$P(1) = 2(1)+1 = 3(1)^2$$
$$3 = 3 \checkmark$$

Assume $P(n)$ is true

$$\sum_{k=1}^{n+1} 2(n+1)+2k-1 = \sum_{k=1}^{n+1} 2n+2+2k-1$$

$$= \sum_{k=1}^{n+1}(2n+2k-1)+2(n+1)$$

$$= \sum_{k=1}^{n+1}(2n+2k-1)+[2n+2(n+1)-1]+2(n-1)$$

$$= \sum_{k=1}^{n+1}(2n+2k-1)+4n+1+2n+2)$$

$$= \sum_{k=1}^{n+1}(2n+2k-1)+6n+3$$

$$= 3n^2+6n+3 = 3(n+1)^2 \quad \square$$

1.12) (a) $n=1$ $\quad (a+b)^1 = \binom{1}{0}a^1 + \binom{1}{1}b^1 = \underline{a+b}$

$n=2$ $\quad (a+b)^2 = \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2 = \underline{a^2+2ab+b^2}$

$n=3$ $\quad (a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$

$= \underline{a^3 + 3a^2b + 3ab^2 + b^3}$

(b) $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

$$\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!}{k(k-1)!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$$

$$= \frac{n!(n-k+1)! + n!(k)(n-k)!}{(n+1-k)!(k-1)!(k)(n-k)!} = \frac{n!(n-k+1+k)}{(n+1-k)!(k-1)!(k)(n-k)}$$

$$= \frac{(n+1)\,n!}{(n+1-k)!(k-1)!(k)(n-k)!} = \frac{(n+1)!}{(n+1-k)!(k!)} = \binom{n+1}{k} \quad \square$$

(c) $P(1)$: seen in part (a)

Assume $P(n)$ to be true

$$P(n+1):\ (a+b)^{n+1} = (a+b)^n \cdot a+b$$

$$= \left[\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n\right](a+b)$$

$$= \left[\binom{n}{0}a^n \cdot a + \binom{n}{1}a^{n-1}b\cdot a + \ldots + \binom{n}{n-1}ab^{n-1}\cdot a + \binom{n}{n}b^n \cdot a\right]$$

$$+ \left[\binom{n}{0}a^n\cdot b + \binom{n}{1}a^{n-1}b\cdot b + \ldots + \binom{n}{n-1}ab^{n-1}\cdot b + \binom{n}{n}b^n\cdot b\right]$$

$$= \binom{n}{0}a^{n+1} + \left[\binom{n}{1}a^n b + \binom{n}{0}a^n b\right] + \left[\binom{n}{2}a^{n-1}b^2 + \binom{n}{1}a^{n-1}b^2\right] +$$

$$\ldots \left[\binom{n}{n}b^n a + \binom{n}{n-1}b^n a\right] + \binom{n}{n}b^{n+1}$$

$$= \binom{n+1}{0}a^{n+1} + \binom{n+1}{1} + \ldots + \binom{n}{n}b^n a + \binom{n+1}{n+1}b^{n+1} \quad \square$$

2.1) Rational Zeros Thm: (RZT)

$x^n + c_{n-1}x^{n-1} + \dots + c_1x + c_0 = 0$ $\quad$ $x^2 - 3 = 0$

$x = \sqrt{3}$

$c_n = 1$, $c_0 = 3$

so the only rational numbers that could possible be the soln to $x^2 - 3$ are $\pm 1, \pm 3$, by RZT. Since $\sqrt{3}$ is the soln, but not amongst the possible rational solns, we can say that $\sqrt{3}$ is irrational

$\sqrt{5}$ : $x^2 - 5 = 0$

RZT: set of solns: $S_1 = \{\pm 1, \pm 5\}$; $\sqrt{5} \notin S_1$

$\therefore \sqrt{5}$ is irrational

$\sqrt{7}$ : $x^2 - 7 = 0$

RZT: set of solns: $S_2 = \{\pm 1, \pm 7\}$; $\sqrt{7} \notin S_2$

$\therefore \sqrt{7}$ is irrational

$\sqrt{24}$ & $\sqrt{31}$

are solved similarly.

2.2) Proof similar to 2.1, using RZT

$\sqrt[3]{2}$ : $x^3 - 2 = 0 \rightarrow x = \sqrt[3]{2}$

RZT: $\pm 1, \pm 2$ solutions $\quad$ $\sqrt[3]{2}$ is not in the set of rational solutions, but is a soln $\therefore$ it is irratio

$\sqrt[7]{5}$ : $x^7 - 5 = 0$

$x = \sqrt[7]{5}$

RZT: $\pm 1, \pm 5$ same logic as $\sqrt[3]{2}$

$\sqrt[4]{13}$ : $x^4 - 13 = 0$

$x = \sqrt[4]{13}$

RZT: $\pm 1, \pm 13$ same logic as $\sqrt[3]{2}$.

2.7) (a) $\sqrt{4+2\sqrt{3}} - \sqrt{3}$ $\qquad 2\sqrt{3} = \sqrt{12}$

$\cancel{\sqrt{4+2\sqrt{12}} - 2\sqrt{8}}$

$x = \sqrt{4+2\sqrt{3}} - \sqrt{3}$

$(x+\sqrt{3})^2 = (\sqrt{4+2\sqrt{3}})^2$

$x^2 + 2x\sqrt{3} + 3 = 4+2\sqrt{3}$

$x^2 - 1 + 2x\sqrt{3} - 2\sqrt{3} = 0$ $\qquad x^2 - 1 = (x+1)(x-1)$

$x^2 - 1 + 2\sqrt{3}(x-1) = 0$

$(x-1)(x+1+2\sqrt{3}) = 0$

$\underline{x = 1, \ -1-2\sqrt{3}}$ $\rightarrow$ $\sqrt{4+2\sqrt{3}} - \sqrt{3}$

cannot produce a negative value b/c $\sqrt{4+2\sqrt{3}} > \sqrt{3}$

$\therefore \underline{\underline{x = 1}}$ $\qquad 1 \in \mathbb{Q}$ ✓ $\square$

(b) $\sqrt{6+4\sqrt{2}} - \sqrt{2} = x$

$(x+\sqrt{2})^2 = (\sqrt{6+4\sqrt{2}})^2$

$x^2 + 2x\sqrt{2} + 2 = 6 + 4\sqrt{2}$

$x^2 + 2x\sqrt{2} - 4 - 4\sqrt{2} = 0$

$x^2 - 4 + 2\sqrt{2}(x-2) = 0$

$(x+2)(x-2) + 2\sqrt{2}(x-2) = 0$

$(x-2)(x+2+2\sqrt{2}) = 0$

$x = 2, \ -2-2\sqrt{2}$ $\rightarrow$ $\sqrt{6+4\sqrt{2}} > \sqrt{2}$

$\therefore x > 0$ & $x = 2$ not $-2-2\sqrt{2}$

$2 \in \mathbb{Q}$ ✓ $\square$

3.6) (a) $|a+b+c| \le |a| + |b| + |c|$ $\quad a,b,c \in \mathbb{R}$

$\triangle\le$: $|a+b| \le |a| + |b|$

$\cancel{X}$ $|(a+b)+c| \le |a+b| + |c|$

$|(a+b)| + |c| \le |a|+|b|+|c|$ $\quad$ if $a \le b$, then $a+c \le b+c$

$|a+b+c| \le |a|+|b|+|c|$ $\square$

I like this proof method →

$|a+b| \le |a| + |b|$

$|a+b| + |c| \le |a|+|b| + |c|$ $\quad$ Triangle ineq.

$|a+b+c| \le |a|+|b|+|c|$ ✓ $\square$ $\quad |a+b| + |c| \ge |a+b+c|$

(b) $P(1)$: $|a_2| \le |a_1|$ ✓

$|a_1 + a_2| \le |a_1| + |a_2|$ ✓ Triangle ineq.

Assume $P(n)$ to be true.

$P(n)$: $|a_1 + a_2 + \ldots + a_n| \le |a_1| + |a_2| + \ldots + |a_n|$

$P(n+1)$: $|a_1 + a_2 + \ldots + a_n| + |a_{n+1}| \le |a_1| + |a_2| + \ldots + |a_n| + |a_{n+1}|$

Triangle Ineq. $\rightarrow |a_1 + a_2 + \ldots + a_n| + |a_{n+1}| \ge |a_1 + a_2 + \ldots + a_{n+1}|$

$\rightarrow |a_1 + a_2 + \ldots a_{n+1}| \le |a_1| + a_2 + \ldots + |a_{n+1}|$

4.11) $a<b$; $a,b \in \mathbb{R}$, show infinitely many rationals between $a$ & $b$.

Proof by induction

$P(1)$: $a < r_1 < b$ ✓ True by Denseness of $\mathbb{Q}$

Assume $P(n)$ to be true, st

$a < r_n < r_{n-1} < \dots < r_1 < b$

by Denseness of $\mathbb{Q}$ we can find some $r_{n+1} \in \mathbb{Q}$ st $a < r_{n+1} < r_n$

$\rightarrow a < r_{n+1} < r_n < \dots < r_1 < b$

$P(n+1)$ holds $\square$

$\boxed{\text{Denseness of } \mathbb{Q}\text{: If } a,b \in \mathbb{R} \text{ \& } a<b\text{, then there exists a rational } r \in \mathbb{Q} \text{ such that } a<r<b.}$

4.14) (a) Prove $\sup(A+B) = \sup(A) + \sup(B)$

For all $a \in A$, $a \leq \sup(A)$

For all $b \in B$, $b \leq \sup(B)$

$\rightarrow$ For all $a+b \in A+B$, $a+b \leq \sup(A) + \sup(B)$

$\sup(A+B) \leq \sup(A) + \sup(B)$

For any $k > 0$, there exists an $a \in A$ st

$a \geq \sup A - k/2$

, there exist a $b \in B$ s t

$b \geq \sup B - k/2$

Thus, $\exists a+b \in A+B$ st $a+b \geq \sup(A) + \sup(B) - k$

$\rightarrow \sup(A+B) \geq \sup A + \sup B - k$

$\therefore \sup(A+B) = \sup(A) + \sup(B)$

(b). $\inf(A+B) = \inf(A) + \inf(B)$

textbook proves $\inf(S) = -\sup(-S)$

$\inf(A+B) = -\sup(-A-B)$

$= -\sup(-A) - \sup(-B)$ part (a)

$= \inf(A) + \inf(B)$ $\square$

9.5) (a) $\sqrt{n^2+1} - n \cdot \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$ $\quad \cancel{n^2}+1-\cancel{n^2} = \frac{1}{\sqrt{n^2+1}+n}$

at $n \to \infty$, $\sqrt{n^2+1}$ becomes irrelevant

$\lim \frac{1}{\sqrt{n^2+1}+n} = \frac{1}{n} = 0$

(b) $\lim \left(\sqrt{n^2+n} - n\right)\frac{\left(\sqrt{n^2+n}+n\right)}{\sqrt{n^2+n}+n} = \frac{\cancel{n^2}+n-\cancel{n^2}}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}}$

$\lim \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{\sqrt{1}+1} = 1/2$

(c) $\lim \sqrt{4n^2+n} - 2n \cdot \frac{\sqrt{4n^2+n}+2n}{\sqrt{4n^2+n}+2n} = \frac{\cancel{4n^2}+n-\cancel{4n^2}}{\sqrt{4n^2+n}+2n}$

$\lim_{n\to\infty} \frac{n}{\sqrt{4n^2+n}+2n} = \frac{1}{\sqrt{4+1/n}+2} = \frac{1}{\sqrt{4}+2} = 1/4$