Homework 10

Ross § 33 (4, 7)

4) Give an example of a function f on $[0,1]$ that is not integrable for which $|f|$ is integrable.

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \qquad |f(x)| = \begin{cases} |x| & x \in \mathbb{Q} \\ |-x| & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$f(x)$ is not integrable. Let P be some partition of $[0,1]$

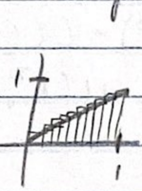
$$P = \{0 = t_0 < t_1 < \dots < t_n = 1\}$$

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}) \underset{=1}{>} 0 \quad \text{clearly}$$

$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}) \underset{=-1}{<} 0 \quad \text{clearly}$$

$\Rightarrow U \neq L$, $f(x)$ is not integrable
$-1 \neq 1$

$|f(x)| = x \in [0,1]$



clearly integrable

$$U(|f(x)|,P) = \sum_{k=1}^{n} M \cdot (t_k - t_{k-1}) = \sum_{k=1}^{n} m \cdot (t_k - t_{k-1}) = L(|f(x)|, P)$$

$1 = 1$ $\square$

7) (a) show $U(f^2,P) - L(f^2,P) \leq 2B[U(f,P) - L(f,P)]$, $\forall P$
$|f(x)| \leq B$

RHS: $2B\left[\sum_{k=1}^{n} M(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}) - \sum_{k=1}^{n} m(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1})\right]$

$2B \cdot D$, D is the difference, $D \geq 0$

LHS: $U(f^2, P) - L(f^2, P)$ $\qquad f(x)^2 - f(y)^2 = [f(x) + f(y)][f(x) - f(y)]$

$$\leq [B + B][f(x) - f(y)]$$
$$= 2B(f(x) - f(y))$$

$$\Rightarrow \sup\{f(x)^2 - f(y)^2\} \leq \sup[2B(f(x) - f(y))]$$
$$\sup\{f(x)^2\} - [f(y)^2] \leq 2B\sup(f(x)) - 2Bf(y)$$
$$M(f^2, P) - f(y)^2 \leq 2B \cdot M(f, P) - 2Bf(y)$$
$$M(f^2, P) - \inf(f(y)^2) \leq 2B \cdot M(f, P) - 2B\inf(f(y))$$
$$M(f^2, P) - m(f^2, P) \leq 2B(M(f, P) - m(f, P))$$
$$U(f^2 P) - L(f^2, P) \leq 2B(U(f, P) - L(f, P)) \quad \square$$

(b) if f is integrable on $[a, b]$, then f^2 is too.

$U(f, P) = L(f, P)$. We know $U(f^2, P) - L(f^2, P) \leq 2B(U(f, P) - L(f, P))$

$$U(f^2, P) - L(f^2, P) \leq 2B \cdot (0)$$
$$u - u \leq 0$$

Note: $U - L \geq 0$ b/c of upper, lower defns.

$$\Rightarrow U(f^2, P) - L(f^2, P) = 0$$
$$U(f^2, P) = L(f^2, P). \quad \square$$

Ross §35

4) $F(t) = \sin t \quad t \in [-\pi/2, \pi/2]$

(a) $\int_0^{\pi/2} x\,dF(x) = \int_0^{\pi/2} x\cos x = fg - \int gf' = x\sin x - \int \sin x\,dx$

$f = x$

$g' = \cos x$

$$= [x\sin x + \cos x]_0^{\pi/2}$$
$$= [\pi/2(1) + 0] - [0 + 1]$$
$$= \underline{\pi/2 - 1}$$

(b) $\int_{-\pi/2}^{\pi/2} x\sin x = x\sin x + \cos x \Big|_{-\pi/2}^{\pi/2}$

$$= [\pi/2(1) + 0] - [-\pi/2(-1) + 0] = \pi/2 - \pi/2 = 0 //$$

9) f cont on $[a,b]$

(a) Show $\int_a^b f\,dF = f(x)[F(b)-F(a)]$ $\quad \forall x \in [a,b]$

Let $M = \sup (a,b)$

$m = \inf (a,b)$

$$m \le f(x) \le M$$

$$\int_a^b m\,dF \le \int_a^b f\,dF \le \int_a^b M\,dF$$

$$m[F(b)-F(a)] \le \int_a^b f\,dF \le M[F(b)-F(a)]$$

$$m \le \frac{\int_a^b f\,dF}{F(b)-F(a)} \le M$$

f is cont. $m \le M$ so by IVT, $f(x) = \dfrac{\int_a^b f\,dF}{F(b)-F(a)}$

$\Rightarrow \int_a^b f\,dF = f(x)[F(b)-F(a)]$ $\square$