Homework 2

Ross

10.9) $s_1 = 1 \qquad s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2$ for $n \geq 1$

(a) $s_2 = \left(\frac{2}{3}\right) \cdot 1 = 2/3$; $s_3 = \left(\frac{3}{4}\right) \cdot \left(\frac{2}{3}\right)^2 = \frac{12}{36} = \frac{1}{3}$

$s_4 = \left(\frac{4}{3}\right)\left(\frac{1}{3}\right)^2 = \frac{4}{45}$

(b) if sequence is monotone → converges

So proof by induction:

$s_1 > s_2$ ✓ part (a)

Assume $P(n)$ to be true

$\leftrightarrow s_n > s_{n+1}$

Now $P(n+1)$: $s_{n+2} = \left(\frac{n+1}{n+2}\right) s_{n+1}^2$

$= \left(\frac{n+1}{n+2}\right) \cdot s_{n+1} \cdot s_{n+1}$

Note $0 \leq s_n \leq 1 \quad \forall n$, so we know then

$0 < \frac{n+1}{n+2} < 1 \quad \forall n$

$\frac{n+1}{n+2} \cdot s_{n+1} \cdot s_{n+1} < s_{n+1}$

$\therefore s_{n+2} < s_{n+1} \quad \forall n$ & the sequence is bounded & monotone thus it converges

(c) $\lim s_{n+1} = \lim \frac{n}{n+1} \cdot s_n^2 = \lim \frac{n}{n+1} \cdot \lim s_n^2$

$= 1 \cdot s^2$

b/c $0 \leq s_n \leq 1$ $\lim s_n^2 \to 0$ or 1, since $s_2 < 1$ & s_n is monotone decreasing, s must $= 0$

$\therefore \lim s_{n+1} = 0$

10.10) $s_1 = 1 \quad s_{n+1} = \frac{1}{3}(s_n + 1) \quad n \geq 1$

(a) $s_2 = 2/3$; $s_3 = 5/9$; $s_4 = 14/27$

(b) Proof by induction: $P(1)$: $s_1 > 1/2$

Assume $P(n)$ to be true (ie $s_n > 1/2$)

$P(n+1)$: $S_{n+1} = \frac{1}{3}(S_n + 1) = \frac{1}{3}(3/2) = 1/2$

$P(n+1)$ is true $\therefore s_n > 1/2 \quad \forall n$.

(c) $s_{n+1} - s_n = 1/3\,(s_n + 1) - s_n$
$= 1/3 - 2/3\, s_n$
$< 1/3 - 2/3 \cdot 1/2 \qquad * s_n > 1/2$
< 0

$\therefore s_{n+1} < s_n \Rightarrow s_n$ is decreasing

(d) $1/2 < s_n \leq 1$ part (c) shows decreasing & bounded
$\rightarrow s_n$ is convergent & $\lim s_n$ exists.

$\lim s_n = \lim s_{n+1}$
$= 1/3 \lim s_n + 1/3$

$\Rightarrow 2/3 \lim s_n = 1/3$
$\lim s_n = 1/2$ $\square$

10.11) $t_1 = 1 \quad t_{n+1} = \left[1 - \frac{1}{4n^2}\right] \cdot t_n \quad n \geq 1$

(a) $t_2 = (1 - 1/16) \cdot 1 = 15/16 \quad t_3 = \left[1 - 1/36\right] \cdot 15/16 = 35/36 \cdot 15/16 \leq 1$

we know that t_n is bounded s.t. $0 < t_n \leq 1 \ \forall n$

$$t_{n+1} - t_n = \left[1 - \frac{1}{4n^2}\right] \cdot t_n - t_n$$

$$= t_n - \frac{t_n}{4n^2} - t_n = t_n\left[1 - \frac{1}{4n^2} - 1\right]$$

$$= t_n\left[\frac{-1}{4n^2}\right] < 0$$

~~thus~~ $t_{n+1} - t_n$ is negative thus, the seq. is monotone decreasing & bounded.

$\therefore$ the lim exists $\square$

(b) it appears that it is approaching 1/2, see t_2, t_3 calculation above

(2) $\forall \varepsilon > 0$, we have $N > 0$ st $\forall n > N$

$|A_n - L| < \varepsilon$, $|B_n - L| < \varepsilon$

then

$a_n \le A_n < L + \varepsilon$ $\quad$ $a_n \ge B_n > L - \varepsilon$

$\Rightarrow L + \varepsilon > a_n > L - \varepsilon \Leftrightarrow |a_n - L| < \varepsilon$ $\square$

$a_n \le b_n \le c_n$ $\quad$ $L = \lim a_n = \lim c_n$

$\forall \varepsilon > 0$, $N > 0$ st $\forall n > N$

$|c_n - L| < \varepsilon$, $|a_n - L| < \varepsilon$

then

$b_n \le c_n < L + \varepsilon$ $\quad$ $c_n \ge b_n > L - \varepsilon$

$\Rightarrow L + \varepsilon > b_n > L - \varepsilon \Leftrightarrow |b_n - L| < \varepsilon$ $\square$