Homework 3

Ross

10.6) (a) Let (S_n) be a sequence st $|s_{n+1} - s_n| < 2^{-n}$ $\forall n \in \mathbb{N}$
Prove S_n is Cauchy.

For a seq. to be Cauchy, it must get arbitrarily close to itself at some tail in the seq.

Think of S_n as a seq (a_n) $a_n = \sum_n^{\infty} \frac{1}{2^n} = \frac{1}{2^{n-1}}$

$\lim a_n = \underline{0}$ so a_n converges to 0. Pick some arbitrary $\varepsilon > 0$. There exists some $n > N \in \mathbb{N}$ where $a_n < \varepsilon$. Pick some $m, n > N$, and $m > n$ st

$$\begin{aligned} |s_m - s_n| &= |s_m - s_{m-1} + s_{m-1} - s_{m-2} + \dots + s_{n+1} - s_n| \\ &\leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \dots |s_{n+1} - s_n| \\ &< \frac{1}{2^{n-m-1}} + \dots + \frac{1}{2^n} \\ &< a_n \\ &< \varepsilon \end{aligned}$$

$\rightarrow \therefore S_n$ is Cauchy $\square$

(b) Yes, it is the same proof w/ $a_n = \sum_n^{\infty} \frac{1}{n}$ which still converges to 0 & the rest of the proof holds. $\square$

11.2) $a_n = (-1)^n$ $b_n = 1/n$ $c_n = n^2$ $d_n = \frac{6n+4}{7n-3}$

(a) a_n : $2n$) ; b_n : $n > 0$; c_n : $n > 0$; d_n : $n > 10$

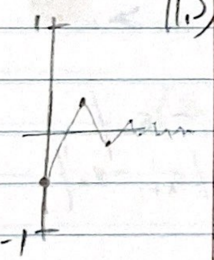
(b) $a_n : \{-1, 1\}$; $b_n : \{0\}$; $c_n : \{+\infty\}$; $d_n : \{6/7\}$

(c) a_n : limsup $= 1$ liminf $= -1$; b_n : limsup $=$ liminf $= 0$; c_n : limsup $=$ liminf $= +\infty$; d_n : limsup $=$ liminf $= 6/7$

(d) a_n : osilates ; $b_n \to 0$, $c_n \to +\infty$; $d_n \to 6/7$

(e) a_n, b_n, d_n bounded ; c_n : unbounded

11.3) $s_n = \cos(n\pi/3)$ $t_n = \frac{3}{4n+1}$ $u_n = \left(-\frac{1}{2}\right)^n$ $v_n = (-1)^n + \frac{1}{n}$



(a) s_n : $6n$ t_n : $n \geq 0$ u_n : $2n$ v_n : $n > 0$

(b) $s_n : \{3n \mid n > 0\}$ $t_n : \{0\}$ $u_n : \{0\}$ $v_n : \{1, -1\}$

(c) s_n : limsup $= 1$ liminf $= -1$ t_n : limsup $=$ liminf $= 0$ u_n : limsup $=$ liminf $= 0$ v_n : limsup $= 1$ liminf $= -1$

(d) $s_n \not\to$; $t_n \to 0$; $u_n \to 0$; $v_n \not\to$

(e) All are bounded.

11.5) let (q_n) be an enumeration of all the rationals in the interval $[0,1]$.

(a) $\boxed{\{r \mid 0 \leq r \leq 1, r \in \mathbb{R}\}.}$
This is true when consider some $\varepsilon > 0$, by denseness of $\mathbb{Q}$, infinitely many rationals $q : r - \varepsilon < q < r$ & q st $r < q < r + \varepsilon$.
0 is included in the set b/c $[0, 0+\varepsilon)$ also has infinitely many reals between them.

(b) $\limsup q_n = \sup\{r / 0 \leq r \leq 1, r \in \mathbb{R}\} = 1$
$\liminf q_n = \inf\{r / 0 \leq r \leq 1, r \in \mathbb{R}\} = 0$ $\square$

(2) Limsup is most easily thought of as the supera of the set of subsequential limits of a sequence.

Limsup is to be thought of as the sup of the tail of a seq., whereas, sup is applied to sets. The limsup of a seq may not be greater than each element in that seq b/c it only applies to the tail of said seq, this can get confusing when sup is used to describe the upperbound, however limsup is not such upperbound, but rather an indicator of tail behavior.