Homework 4

Ross 12

(10) Prove (s_n) is bounded iff $\limsup |s_n| < +\infty$

"$\Rightarrow$" If (s_n) is bounded, we ~~know~~ know that for $\forall n$, $\exists x_1, x_2$

$-\infty < x_1, x_2 < +\infty$ st $x_1 \le s_n \le x_2$. Now $\limsup |s_n|$ is the sup {all subseq. limits in s_n}. Now if we know $s_n \le x_2$, the sequence can thus never exceed that value, only get close to it st $\forall \varepsilon > 0, \exists N$ st $n > N$,

$|s_n - x_2| < \varepsilon$. This means it can at most approach x_2, but never exceed it, thus sup{|subseq. limits|} $= x_2$ or if $|x_1| > |x_2|$, the sequence could approach x_1, but never exceed it thus sup{|subseq. limits|} $= x_1$. Both $|x_1|$ & $|x_2| < +\infty$, thus $\limsup |s_n| < +\infty$

"$\Leftarrow$" If $\limsup |s_n| < +\infty$, $\exists x$, $x \in \mathbb{R}$ st $\limsup |s_n| = x$

Now we have a seq. $A_N = \sup\{|s_n| : n > N\}$ converges to x, there exists N_1 st $|\sup\{|s_n| : n > N_1\} - M| < 1$

$\sup\{|s_n| : n > N_1\} < M+1$

$\rightarrow |s_n| < M+1$

Now construct a set U st $U = \max\{|s_1|, |s_2|, \ldots, |s_N|, M+1\}$

$\forall n$, $|s_n| \le U$

$\therefore |s_n|$ is bounded $\square$

(12) (a) $\sigma_n = \frac{1}{n}\sum_{i=1}^{n} s_i$ Show $\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n$

$M, N \in \mathbb{R}$, for $M > N$, $n > M$:

Showing $\limsup \sigma_n \leq \limsup s_n$

$$\sigma_n = \frac{1}{n}(s_1+s_2+\dots+s_N) + \frac{1}{n}(s_{N+1}+\dots+s_n)$$
$$\leq \frac{1}{M}(s_1+s_2+\dots+s_N) + \frac{1}{n}(s_{N+1}+\dots+s_n)$$
$$\leq \frac{1}{M}(s_1+s_2+\dots+s_N) + \frac{n-N}{n}\sup\{s_n : n > N\}$$
$$\leq 1/M\,(s_1+s_2+\dots+s_N) + \sup\{s_n : n > N\}$$

Now lets take the limit as $M \to \infty$ to find the tail behavior of σ_n.

$$\lim_{M\to\infty} \sigma_M = 0\,\underbrace{\cancel{(s_1+s_2+\dots+s_N)}}_{\to 0} + \sup\{s_n : n > N\}$$

$\to \limsup \sigma_n = \sup\{s_n : n > N\}$ $\to \limsup \sigma_n \leq \limsup s_n$.

We will show $\liminf s_n \leq \liminf \sigma_n$

Let $a_n = -\sigma_n$ & $b_n = -s_n$.

Note then $\limsup a_n \leq \limsup b_n$

$-\limsup a_n \geq -\limsup b_n$

$-\limsup(-\sigma_n) \geq -\limsup(-s_n)$

$\liminf \sigma_n \geq \liminf s_n$

true trivially

$\to \liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n$ $\square$

(b) Let $\lim s_n = t$ s.t. $\forall \varepsilon > 0, \exists N$, s.t. $|s_n - t| < \varepsilon$.

Now if s_n converges, that means all subseq. limits of s_n must converge to the same value, t. This means $\liminf s_n = \limsup s_n = \lim s_n = t$ //

Now since $\liminf s_n \leq \limsup \sigma_n \leq \limsup s_n$

$t \leq \limsup \sigma_n \leq t$

$\to \limsup \sigma_n = t$ $\square$

(c) Let $s_n = (-1, 1, -1, 1, \dots)$ $\sigma_n \to 0$.
$\sigma_n = \frac{1}{n}(0)$ by $s_n \not\to$.

Ross 14

(2) (a) $\sum \frac{n-1}{n^2}$ $a_n = \frac{n-1}{n^2}$ $\lim a_n = 0$, $\sum \frac{n-1}{n^2}$ converges

(b) $\sum (-1)^n$ $\lim (-1)^n \neq 0$, diverges.

(c) $\sum 3n/n^3$ $\lim \frac{3n}{n^3} = \lim \frac{3}{3n^2} = 0$ // converges.

(d) $\sum n^3/3n$ $\lim n^3/3n = \lim 3n^2/3 = +\infty$ diverges.

(e) $\sum \frac{n^2}{n!}$ $\lim n^2/n! = 0$ converges.

(f) $\sum \frac{1}{n^n}$ $\lim \frac{1}{n^n} = \frac{\lim 1}{\lim n^n} = 0$ converges.

(g) $\sum \frac{n}{2^n}$ $\lim \left(\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} \right) = \lim \left(\frac{n+1}{\cancel{2^{n+1}}} \cdot \frac{\cancel{2^n}}{2n} \right) = 1/2 < 1$. converges

(10) ① Root Test: $\limsup |a_n|^{1/n} = r$ ② Ratio Test: $\limsup \frac{|a_{n+1}|}{|a_n|} = r$
$r \geq 1$ $r = 1$

$a_n = \sum 2^{(-1)^n n}$ ① $\limsup |2^{(-1)^n n}|^{1/n} = \limsup |2^{(-1)^n}| = 2$

② $\limsup \frac{|a_{n+1}|}{|a_n|} = \frac{2^{-1 \cdot n}}{2^{1 \cdot n}} = \frac{1/2^n}{2^n} = \frac{1}{\cancel{2^n}} \cdot \frac{\cancel{2^n}}{1} = 1$

Same goes if $a_{n+1} = 2^{1 \cdot n}$
$a_n = 2^{-1 \cdot n}$ $\square$

Rudin Chapter 3

6) (a) $\sqrt{n+1} - \sqrt{n} = a_n$

$\lim a_n = \lim (n+1)^{1/2} - \lim (n)^{1/2} = 0$ converges

(b) $\frac{\sqrt{n+1} - \sqrt{n}}{n} = a_n$

$\lim a_n = 0$ converges ✓

(c) $a_n = (\sqrt[n]{n} - 1)^n$; $\lim a_n = \lim (n^{1/n} - 1)^n = (0-1)^n = 1$

diverges ///

(d) $a_n = \frac{1}{1+z^n}$ for complex values

$\lim a_n \neq 0 \Rightarrow \sum a_n$ diverges.

7) If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. $\forall \varepsilon > 0, \exists N, n > N$ s.t. $|a_n - 0| < \varepsilon$.

Now $\frac{\sqrt{a_n}}{n}$ is a new seq. $\sqrt{a_n} \leq a_n \quad \forall n > 0$

$\frac{a_n}{n} \leq a_n \quad \forall n > 0$

thus we can say $\frac{\sqrt{a_n}}{n} \leq a_n$. Since we know

$\lim a_n = 0$, & $a_n \geq 0$, thus we can write

$0 \leq \frac{\sqrt{a_n}}{n} \leq a_n \rightarrow \lim 0 \leq \lim \frac{\sqrt{a_n}}{n} \leq \lim a_n$

$\rightarrow 0 \leq \lim \frac{\sqrt{a_n}}{n} \leq 0$

$\Rightarrow \lim \frac{\sqrt{a_n}}{n} = 0$ & $\sum \frac{\sqrt{a_n}}{n}$ converges $\square$

9) (a) $\sum n^3 z^n$ $R = \frac{1}{\limsup |a_n|^{1/n}} = \frac{1}{\limsup |n^{3/n} z|} = \frac{1}{1} = \underline{\underline{1}}$ ✓

(b) $\sum \frac{2^n}{n!} z^n$ $R = \frac{1}{\limsup |a_n|^{1/n}} = \frac{1}{\limsup |\frac{2}{n!} z|} = \frac{1}{0} = +\infty_{//}$ ✓

(c) $\sum \frac{2^n}{n^2} z^n$ $R = \frac{1}{\limsup |\frac{2}{n^{2n}} z|} = \frac{1}{2}_{//}$

(d) $\sum \frac{n^3}{3^n} z^n$ $R = \frac{1}{\limsup |\frac{n^{3n}}{3} \cdot z|} = \frac{1}{\infty} = 0_{//}$

11) $\sum \frac{a_n}{1+a_n}$ $\limsup \left|\frac{a_n}{1+a_n}\right|^{1/n} =$