Homework 5

Ross §13

3) (a) $B = \{\text{bounded seqs } x\}$

$d(x,y) = \sup\{|x_j - y_j| : j = 1, 2, \ldots\}$

(1) $d(x,x) = 0$, clear. $|x_j - x_j| = 0$

$d(x,y) > 0$, clear by absolute value. $|a-b| > 0$ some $a \neq b$.

(2) $d(x,y) = d(y,x)$, clear again by abs. value $|a-b| = |b-a|$

(3) $d(x,z) \leq d(x,y) + d(y,z)$

Consider some $N, M \in \mathbb{R}$ st $|x_j| \leq N$, $|y_j| \leq M$

this is true b/c x_j & y_j are bounded.

$$|x_j - y_j| \leq |x_j| + |y_j|$$
$$\leq N + M = X, \; X \in \mathbb{R}$$
$$|x_j - z_j| \leq |x_j - y_j| + |y_j - z_j|$$
$$\sup|x_j - z_j| \leq \sup|x_j - y_j| + \sup|y_j - z_j|$$
$$d(x,z) \leq d(x,y) + d(y,z)$$

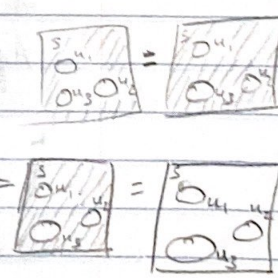
(b) $d'(x,y) = \sum_{j=1}^{\infty} |x_j - y_j|$ is not a metric

Consider $x_j = 1, 1, 1, 1 \ldots$

$y_j = 0, 0, 0, 0, \ldots$

$d'(x,y) = +\infty$ which is infinite, thus not a metric

5) (a) $\bigcap\{S \setminus U : U \in \mathcal{U}'\} = S \setminus \bigcup\{U : U \in \mathcal{U}'\}$



$$S \setminus \bigcup\{U : U \in \mathcal{U}'\} = S \cap \{\bigcup U : U \in \overline{\mathcal{U}}'\}$$
$$= S \cap \{\bigcap \overline{U} : U \in \mathcal{U}'\}$$
$$= \bigcap\{S \cap \overline{U} : U \in \mathcal{U}'\}$$
$$= \bigcap\{S \setminus U : U \in \mathcal{U}'\} \; \square$$

(b) $\{K_\alpha\}$ is a collection of closed sets
then $U_\alpha = \overline{\{K_\alpha\}}$ is open.

$\bigcup U_\alpha$ is open & its complement $\overline{(\bigcup U_\alpha)}$ is closed

$$\overline{(\bigcup U_\alpha)} = \bigcap(\overline{U_\alpha}) = \bigcap\{K_\alpha\}$$

So the collection of any # of closed sets is closed.

$\square$

7) Every open set in $\mathbb{R}$ is the disjoint union of a finite or infinite sequence of open intervals.

Let $E \subseteq \mathbb{R}$ be an open set in $\mathbb{R}$. Let x_n be an enumeration of rationals in E.
Let $a_n = \inf\{a \in \mathbb{R} : (a, x_n] \subseteq E\}$
$b_n = \sup\{b \in \mathbb{R} : [x_n, b) \subseteq E\}$

let $p \in E$. There exists $r > 0$ s.t. $A = \{s : d(s,p) < r\} \subseteq E$
By denseness of $\mathbb{Q}$, there exists $x_i \in A$, implies $x_i \in (a_i, b_i)$
Thus $E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$.

$$E = \bigcup_{n=1}^{\infty} .$$

To show (a_n, b_n) are disjoint. Assume for contradiction that $A = (a_n, b_n) \cap (a_m, b_m) \neq \emptyset$ $n \neq m$.
Again $x_i \in A$.

$a_n = a_i$ for all $a_n < x_i \leq b_n$
$a_m = a_i$ for all $a_m < x_i \leq b_m$
$\rightarrow a_n = a_m$

similarly $b_n = b_i$ for all $b_n < x_i \leq b_n$
$b_m = b_i$ for all $b_m < x_i \leq b_m$
$\rightarrow b_n = b_m$

Thus (a_n, b_n) are disjoint. $\square$

3) (13.7)

~~Show every open set in $\mathbb{R}$ is the disjoint union of a finite or infinite sequence of open intervals.~~
~~We will show it can only be the union of open intervals.~~
~~Consider intervals of form $[a,b)$, $(a,b]$, $[a,b]$. There doesn't exist $B_r(a)$ st $r>0$, $a-r=a$ for the interval $[a,b)$. Similar argument can be made for the other intervals.~~
~~Now we know every open set must be comprised only of disjoint union of open intervals $\square$~~

4) metric space: (X,d), $S \subseteq X$, $\overline{S} = \{p \in X : \text{there is a subseq } (p_n) \text{ in } S,\ p_n \to p\}$

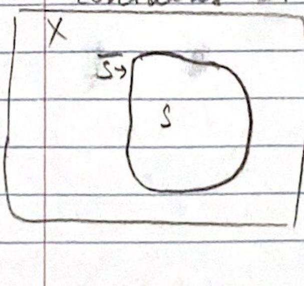
Prove that taking closure again won't make it bigger.
i.e $S_1 = \overline{S}$, $S_2 = \overline{S_1}$ then $S_1 = S_2$. $\overline{S} = \overline{\overline{S}}$.
$\overline{S} \subseteq \overline{\overline{S}}$, so we will show $\overline{\overline{S}} \subseteq \overline{S}$ to show $\overline{S} = \overline{\overline{S}}$.
$\overline{\overline{S}} = \{\overline{p} \in \overline{S} : \text{there is a subseq. } (\overline{p_n}) \text{ in } \overline{S}\ \overline{p_n} \to \overline{p}\}$

Take some $\overline{p} \in \overline{\overline{S}}$, there is then a seq. $(\overline{p_n})$ w/ $\overline{p_i}$ in $\overline{S}$ st $\overline{p_n} \to \overline{p}$, want to show $\overline{p}$ is in fact in $\overline{S}$.

Let $\varepsilon > 0$. $\exists N_1$ st $\forall n_1 > N_1$, $|\overline{p_{n_1}} - \overline{p}| < \varepsilon$
Fix the n_1. Now $\overline{p_{n_1}} \in \overline{S}$ st $\exists N$ st $\forall n_2 > N$ $|p_{n_2} - \overline{p_{n_1}}| < \varepsilon$

Thus $\forall n_2 > N$, $|p_{n_2} - \overline{p}| \leq |p_{n_2} - \overline{p_{n_1}}| + |\overline{p_{n_1}} - \overline{p}| < \varepsilon + \varepsilon < 2\varepsilon$
$\to$ there exists a seq. $(p_n) \in X$ st $p_n \to p$.

5) Prove that $\overline{S}$ is the intersection of all closed subsets in X that contains S.



WTS: $\overline{S} \subset \bigcap \{F \subset X \text{ closed}, S \subset F\}$
Let $s \in \overline{S}$. Then exists a seq $(s_n) \in S$ st $s_n \to s$
Pick some F st $S \subset F$ & F is closed. B/c F is closed, $\forall (f_n) \in F$, $(f_n) \to f$, $f \in F$.
So b/c $s_n \in F$, $s \in F$. ~~Now~~

$\Rightarrow$ over

F is arbitrary, so we can say $s \in \cap \{F \subset X \text{ closed}, S \subset F\}$
b/c they all meet the same criteria.
Now. Set $\bar{S} = F$, we know $\bar{S}$ is closed & $S \subset \bar{S}$ by defn.
$\therefore \bar{S} = \cap \{F \subset X \text{ closed}, S \subset F\}$.