Homework 6

Axler §3E

1) (1) $T: V \to W$, graph $T = \{(v, Tv) \in V \times W : v \in V\}$

Prove T is linear iff the graph of T is a subspace of $V \times W$

"$\Rightarrow$": T is linear, wts graph of T is a subspace of $V \times W$

need: (1) closed under addition + scalar mult.

(2) contains zero vector

(1) $(v_1, Tv_1), (v_2, Tv_2) \in$ graph of T $\quad v_1, v_2 \in V$

$= a(v_1, Tv_1), b(v_2, Tv_2)$ $\quad a, b \in \mathbb{F}$

$= (av_1, aTv_1) + (bv_2, bTv_2)$

$= (av_1 + bv_2, aTv_1 + bTv_2)$

$= (av_1 + bv_2, T(av_1 + bv_2)) \in$ graph of T $\therefore$ closed ✓

(2) let $v_0 = 0$ & $\ker T = \{0\}$

$(v_0, Tv_0) = (0, 0)$ ✓

"$\Leftarrow$": the graph of T is a subspace of $V \times W$, wts T is linear.

need: $T(av_1 + bv_2) = aTv_1 + bTv_2$

$a(v_1, Tv_1) + b(v_2, Tv_2) \in$ graph of T

$(av_1, aTv_1) + (bv_2, bTv_2)$

$\Rightarrow (av_1 + bv_2, aT(v_1) + bT(v_2))$

$\Rightarrow (av_1 + bv_2, T(av_1 + bv_2)) \in$ graph of T

Hence $aTv_1 + bTv_2 = T(av_1 + bv_2)$ and T is linear $\square$

2) (11) Suppose $v_1, —, v_m \in V$. Let $A = \{\lambda_1 v_1 + \dots + \lambda_m v_m : \lambda_1, —, \lambda_m \in \mathbb{F}$ & $\lambda_1 + \_ + \lambda_m \in 1\}$

(a) Prove A is an affine subset

Fix any $w \in A$

$B = \{v - w : v \in A\} \Rightarrow A = w + B$

if $a \in A$ then $a - w \in B$

$(a-w) + w \in A$

so now will show that B is a subspace.

conditions: (1) zero vector: When $v = w$, $v - w = 0$, $0 \in B$

(2) "+": $b_1, b_2 \in B$

$b_1 = a_1 - w \quad \forall a_1 \in A$; $b_2 = a_2 - w \quad \forall a_2 \in A$

$$b_1 + b_2 = a_1 - w + a_2 - w$$
$$= (\sum \lambda_i v_i - w) + (\sum \delta_i v_i - w)$$
$$= (\sum \lambda_i v_i - \sum \delta_i v_i) + (\sum \delta_i v_i - \sum \delta_i v_i)$$
$$= (\sum \lambda_i v_i - \sum \delta_i v_i + \sum \delta_i v_i) - (\sum \delta_i v_i)$$

" 1   -1   1   — w "

= something in A $- w$ ✓

(3) "·": $\alpha \in \mathbb{F}$, $b_1 \in B$

$\alpha b_1 = \alpha a_1 - \alpha w \in B$

$$\alpha b_1 = \alpha(a_1 - w) = \alpha(\sum \lambda_i v_i - \sum \delta_i v_i)$$
$$= \alpha \sum \lambda_i v_i - \alpha \sum \delta_i v_i = \alpha \lambda_1 v_1 + \dots + \alpha \lambda_m v_m - \alpha \delta_1 v_1 \dots - \alpha \delta_m v_m$$
$$= (\alpha \lambda_1 v_1 - \alpha \delta_1 v_1) + (\alpha \lambda_2 v_2 - \alpha \delta_2 v_2) + \dots + (\alpha \lambda_m v_m - \alpha \delta_m v_m)$$
$$= \sum \alpha \lambda_i v_i - \sum \alpha \delta_i v_i = \alpha \sum v_i - \alpha \sum v_i = \alpha - \alpha = 0 \in B \checkmark$$

(b) $\{v_1, —, v_m\} \subseteq v + U \Longrightarrow A \subseteq v + U$.

* affine subsets are either identical or disjoint. *

WTS: $(a - v) \in U$

- $a \in A$ so $a = \lambda_1 v_1 + \dots + \lambda_m v_m$
- $v_i \in v + U$, so $v_i = v + u_i$
- $a - v = (\lambda_1 v_1 + \_ + \lambda_m v_m) - v$

$$= \lambda_1(v + u_1) + \lambda_2(v + u_2) + \_ + \lambda_m(v + u_m) - v$$
$$= \lambda_1 v + \lambda_1 u_1 + \lambda_2 v + \lambda_2 u_2 + — + \lambda_m v + \lambda_m u_m - v$$
$$= \underbrace{(\lambda_1 v + \dots + \lambda_m v - 1 \cdot v)}_{0} + (\lambda_1 u_1 + \dots + \lambda_m u_m)$$

$0 + u \in U$ Hence $a - v \in U$. □

$a \in A$

$a = v + u$ some $u \in U$

$a - v = u$

so $v + (a - v) = a$

$v + u : u \in U$

2) (II)

(c) There is a v & a U st:

(i) $A = v + U$

(ii) $\dim(U) \leq m-1$

$$U = \{ v - \overset{\in A}{v_1} : v \in A \}$$

$$v_1 + U = A$$

$$A = \{ \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m \mid \Sigma \lambda_i = 1 \}$$
$$= \{ (1 - \lambda_2 - \dots - \lambda_m) v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m \mid \lambda_2, \dots, \lambda_m \in \mathbb{F} \}$$
$$= \{ v_1 - \lambda_2 v_1 - \dots - \lambda_m v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m \mid \lambda_2, \dots, \lambda_m \in \mathbb{F} \}$$
$$= \{ v_1 - \lambda_2 (v_2 - v_1) - \dots - \lambda_m (v_m - v_1) \mid \lambda_2, \dots, \lambda_m \in \mathbb{F} \}$$
$$= v_1 - \operatorname{span}(v_2 - v_1, \dots, v_m - v_1) = v_1 + U$$

U is the subspace defined by $\operatorname{span}(v_2 - v_1, \dots, v_m - v_1)$. B/c this is a spanning list of U, & any spanning list of U can be reduced to a basis. We know that the length of the basis of U must be less than or equal to the length of our spanning list.

Hence $\dim(U) \leq m-1$. $\square$

3)

(20) Suppose U is a subspace of V. Define $T': \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ by
$T'(S) = S \circ \pi$

(a) Show T' is linear. WTS: closed under "+" & "·".

$S, T \in \mathcal{L}(V/U, W)$, $a, b \in \mathbb{F}$ & $v \in V$.

$$T'(aS + bT) = (aS + bT)(\pi(v))$$
$$= (aS + bT)(v + U) = (aS)(v+U) + (bT)(v+U)$$
$$= a(S(v+U)) + b(T(v+U))$$
$$= a(S \circ \pi(v)) + b(T \circ \pi(v))$$

Hence $T'(aS + bT) = a(S \circ \pi(v)) + b(T \circ \pi(v))$ ✓

(b) T' is inj if $\ker(T') = \{0\}$

$T'(S) = 0$ means $S \circ \pi$ is the zero lin. map on V.

$\forall v \in V$ $S \circ \pi(v) = 0$

For any $v + U \in V/U$, $S(v+U) = S \circ \pi(v) = 0 \to S$ is the zero lin. map on V/U.

Thus $T'(S) = 0 \to S = 0$ $\therefore T$ is inj. $\square$

(c) Show range $T' = \{T \in \mathcal{L}(V, W) : Tu = 0 \text{ for every } u \in U\}$.

If $x \in \operatorname{im} T'$, $\exists S \in \mathcal{L}(V/U, W)$ st $T'(S) = x \Rightarrow x \in \operatorname{im} T'$ iff $x = S \circ \pi$, $S \in \mathcal{L}(V/U$

$\operatorname{im} T' = \{S \circ \pi \mid S \in \mathcal{L}(V/U, W)\}$. Now if $x \in \operatorname{im} T'$ & $u \in U$,

then $x(u) = S \circ \pi(u) = S(u) = 0$. Thus $x \in \operatorname{im} T'$, then $x(u) = 0$ $\forall u \in U$

Now, let $x \in \mathcal{L}(V, W)$ st $x(u) = 0$ still. We will show $\exists S \in \mathcal{L}(V/U, W)$ st $x = S \circ \pi$.

Let $v + U \in V/U$ & $S(v+U) = x(v)$. WTS S is well defined + lin.

$v + U = w + U \to v - w \in U \to x(v - w) = 0. \to x(v) = x(w)$

$\to S(v+U) = S(w+U) \therefore S$ is well defined.

$a, b \in \mathbb{F}$.

$$S(a(v+U) + b(w+U)) = S(av + aU + bw + bU)$$
$$= S(av + U + bw + U)$$
$$= S(av + bw + U) \checkmark \text{ linear}$$

Hence if $x(u) = 0$ $\forall u \in U$, $\exists S \in \mathcal{L}(V/U, W)$ st $x = S \circ \pi$. $x \in \operatorname{im} T'$.

& $x \in \operatorname{im} T'$ iff $x = S \circ T$, $S \in \mathcal{L}(V/U, W) \Rightarrow \operatorname{im} T' = \{x \in \mathcal{L}(V, W) \mid x(u) = 0 \ \forall u \in U\}$ $\square$

4) Axler §3F

(8) (a) Show $(1, x-5, \ldots, (x-5)^m)$ is a basis of $P_m(\mathbb{R})$

Proof by induction:

Base Case: $m=0$. $(x-5)^0 = 1$ ✓

$1 = 1$ ✓

Inductive step: Assume $1, x-5, \ldots, (x-5)^{m-1}$ is a basis of $P_{m-1}(\mathbb{R})$

WTS: $1, x-5, \ldots, (x-5)^m$ is a basis of $P_m(\mathbb{R})$.

First will show LI: $a + b(x-5) + \ldots + m(x-5)^m = 0$

$\Rightarrow a + bx - 5b + cx^2 + 2cx + 25c + \ldots + mx^m + \ldots + m5^m = 0$

after combining like terms $\Rightarrow mx^m + \alpha_{m-1}x^{m-1} + \ldots + \alpha_1 x + \alpha_0 = 0$

$x^m, \ldots, x, 1$ are lin ind. by std. basis of $P_m(\mathbb{R})$ $\alpha_i \in \mathbb{F}$ $\forall i$

$\Rightarrow m = \alpha_i = 0$ $\forall i, 1 \leq i \leq m-1$.

$\therefore a + b(x-5) + \ldots + m(x-5)^m = 0$ when $a = b = \ldots = m = 0$

Hence $(1, x-5, \ldots, (x-5)^m)$ is LI.

Now that we know our list of length $m+1$ is LI & $\dim P_m(\mathbb{R}) = m+1$
so we can say that the dim of a fin dim vector space
is the length of every basis.

Hence $(1, x-5, \ldots, (x-5)^m)$ is a basis of $P_m(\mathbb{R})$.

(b) Dual space: $\phi: P_m(\mathbb{R}) \to \mathbb{R}$

$\phi(p) \longmapsto p(6) \in V^*$ "evaluate every polynomial @ 6"

Dual basis: $\phi_i(p_j) = \{ {1 \atop 0} {i=j \atop i \neq j}$

basis $= \{\phi_1, \ldots, \phi_m\}$ where $\phi_i(a + b(x-5) + \ldots + m(x-5)^m) = i$

$= \{a, b, \ldots, m\}$ is a dual basis of $(1, x-5, \ldots, (x-5)^m)$