Homework 7

1) X, Y are open cover cpt. $\cdot$ So, let $Z = (X \times Y)$

$X \subset \bigcup_{\alpha \in I_X} U_\alpha$ U_α is open

I_X, I_Y are finite $\quad Z \subset \left[\bigcup_{\alpha \in I_X} U_\alpha\right] \times \left[\bigcup_{\beta \in I_Y} U_\beta\right]$

$Y \subset \bigcup_{\beta \in I_Y} U_\beta$ U_β is open.

$\forall z \in Z, \quad z \subset \underbrace{\left[\bigcup_{\alpha \in I_X} U_\alpha\right]}_{\text{finite union}} \times \underbrace{\left[\bigcup_{\beta \in I_Y} U_\beta\right]}_{\text{finite union}}$

ie. finite union $\times$ finite union $\Rightarrow$ finite union

$a \cup b \cup c = \{a, b, c\}$

$d \cup e \cup f = \{d, e, f\}$ $\{a, b, c\} \times \{d, e, f\} = \{bf - ce, cd - af, ae - bd\}$?

$Z \subset$ union of finite subcover. $\Rightarrow Z$ is open cover cpt $\square$

2) $f: X \to Y$, f is cont. $A \subset X$.

(a) False, $A = (0, 1)$, open in $\mathbb{R}$, $X = Y = \mathbb{R}$,
$f(x) = 0$.
$f(a) = 0$, $\forall a \in A$. 0 itself is closed b/c any seq. created will converge to 0, hence $f(A)$ is not open $\square$

(b) False, $A = (0, 1) = X$, $Y = \mathbb{R}$, $f(x) = x$
A is closed in X, therefore it is closed; however, it is not closed in $\mathbb{R}$, or Y. $f(A) = (0, 1)$ in $\mathbb{R}$ is open. Hence $f(A)$ is open when A is closed $\rightarrow\leftarrow$ $\square$

(c) False, let $A = (0, 1)$, $X = Y = \mathbb{R}$, $f(x) = (1/x)$.
A is bounded, but $f(A) = (1, \infty)$ which is unbounded.
$\rightarrow\leftarrow$ $\square$

(d) True, If A is compact, $\forall (a_n)$ in A, $\exists (a_{n_k}) \to a \in A$.
We know f is continuous & by Thm 2, §2, Pugh,
if f is continuous then:

$$\lim (a_{n_k}) = a$$
$$\lim f(a_{n_k}) = f(a).$$

So, every seq in $f(A)$ converges to a point in $f(A)$ Hence $f(A)$ is seq. compact $\square$

(e) True, Assume for contradiction that A is connected, but $f(A)$ is not. Then $f(A) = G \sqcup H$ where G, H are open, non empty.

$\to f^{-1}(G)$, $f^{-1}(H)$ would also be open. b/c f is conti
Then $A = f^{-1}(G) \sqcup f^{-1}(H)$, this would mean that A is disconnected $\to\leftarrow$. $\square$

3) $[0,1]$ is bounded & closed in $\mathbb{R}$, therefore it is compact. Q2(d) showed then that $f([0,1])$ must be compact $\to \mathbb{R}$ is compact. This is false b/c $\mathbb{R}$ is unbounded. Hence there is no such function f. $\square$