Homework 8

1) Show $f_n(x) = \frac{n+\sin x}{2n+\cos(n^2x)}$ uniformly conv.

uniform conv: $f_n \to f$ if $\forall \varepsilon > 0$, $\exists N > 0$ st $\forall n > N$, $\forall x \in (a,b)$ $|f_n - f| < \varepsilon$.

Clearly $f_n \to \frac{1}{2}$. $\lim_{n\to\infty} f_n(x) = \lim \frac{n + \sin(x)}{2n + \cos(n^2x)}$ ($\sin(x) \to [-1,1]$, $\cos(n^2x) \to [-1,1]$)

$= \lim \frac{n}{2n} = \frac{1}{2}$ //

$$\left| f_n(x) - \frac{1}{2} \right| = \left| \frac{n+\sin x}{2n+\cos(n^2x)} - \frac{1}{2} \right| = \left| \frac{\cancel{2n} + 2\sin x - \cancel{2n} - \cos(n^2x)}{4n + 2\cos(n^2x)} \right|$$

$$\leq \left| \frac{2-(-1)}{4n-2} \right| = \left| \frac{3}{4n-2} \right|$$

Pick N st $n > N$, $\varepsilon < \left|\frac{3}{4n-2}\right|$. Thus $f_n \xrightarrow{\text{uniformly}} 1/2$ □

2) $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Show $f(x)$ is cont. on $[-1,1]$ if $\sum |a_n| < \infty$.

Prove $\sum n^{-2} x^n$ is cont on $[-1,1]$

$f_n(x) = a_n x^n \Rightarrow |f_n(x)| = |a_n x^n| = |a_n| \cdot |x^n|$
$\leq |a_n|$ by ($\forall x \in [-1,1]$)

Thus $|f_n(x)| \leq |a_n|$

& $\sum |a_n| < \infty = M_n$

so $|f_n(x)| \leq |a_n|$ & $\sum |a_n| < \infty$, thus by Weierstrauss M-test $\sum f_n(x)$ conv. uniformly.

By 7.12 Rudin, $f(x)$ is cont on $[-1,1]$

<u>Note:</u> $\underset{a_n x^n}{\overset{\shortparallel}{f_n(x)}}$ on $[-1,1]$ is clearly cont.

2) Prove $\sum n^{-2}x^n$ is cont on $[-1,1]$

$f_n(x) = n^{-2}x^n \quad |f_n(x)| = |n^{-2}| \cdot |x^n|$
$\leq |n^{-2}|$

$$\sum_{n=1}^{\infty} n^{-2} = \sum \frac{1}{n^2}$$

which is conv. by Power Test, $2>1$
Thus $\sum f_n(x)$ is uniformly conv. Since $f_n(x) = \frac{x^n}{n^2}$ is

cont. over $[-1,1]$, using Rudin 7.12 again, $f(x)$ is cont on $[-1,1]$ $\square$.

3) $f(x) = \sum x^n$ on $(-1,1)$, $f(x)$ is cont, but not uniformly conv.

Weierstrauss M-test: Suppose $\{f_n\}$ is a seq of functions on E, $|f_n(x)| \leq M_n$
then $\sum f_n(x)$ is uni conv. if $\sum M_n$ conv.

$|f_n(x)| = |x_n| = |1| \cdot |x^n|$ $f(x)$ to $[a,a]$
Pick some a, $0<a<1$, fix $x=a$, $\Rightarrow |f_n(x)| \leq |a^n| < 1$

$\sum |a_n| = \sum \frac{1}{1-a} < \infty$, $a<1$, hence this conv, & $f(x)$ is

continuous on $[-a,a]$. Now for unif. conv. part, let
$f_n \to f$ st $f_n = \sum_{i=1}^{n} x^i$, $f = \sum_n^{\infty} x^n$

$$|f_n - f| = \left|\sum x^i - \sum x^n\right| = \left|\frac{x(1-x)}{1-x} - \frac{x}{(1-x)}\right| = \left|\frac{x^{n+1}}{x+1}\right|$$

Pick $x = 1/2$, $\forall N>0$, $n>N$.

$$\Rightarrow \left|\frac{(1/2)^{n+1}}{1/2+1}\right| \leq \left|\frac{(1/4)}{3/2}\right| = \left|\frac{1}{6}\right|. \quad \text{Pick } \varepsilon = 1/7$$

$|f_n - f| > \varepsilon$, $1/6 > 1/7 \Rightarrow$ Hence f is not uniformly convergent $\square$