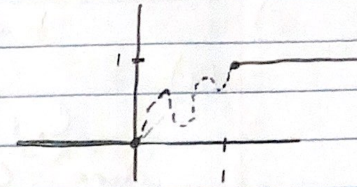
Homework 9

1) $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = \begin{cases} 0 & x \le 0 \\ 1 & x \ge 1 \\ [0,1] & \text{other} \end{cases}$



f is smooth, $f \in C^\infty(\mathbb{R})$

Need some func that $f^{(n)}(x) = 0$, consider $e^{-1/x}$ which is infinitely differentiable, by Ex 3 - Ross, & $f^{(n)}(x) = 0$. $x > 0$.

Now, need to make $e^{-1/x}$ get to 1 at $x = 1$:

Let $f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0 \end{cases}$

This is infinitely diff., consider $f(1-x)$, it is also infinitely diff. $\frac{d^n}{dx^n} f(1-x) = (-1)^n f(1-x)$

$f(1-x) = 0 \quad x \ge 1$

$f(1-x) = e^{-1/x} \quad x < 1$

Hence, $f(x) + f(1-x) > 0$, $\forall x$.

$x < 1$, $f(1-x) > 0$, $f(x) \ge 0$

$x \ge 1$, $f(1-x) = 0$, $f(x) > 0$.

let $g(x) = \dfrac{f(x)}{f(x) + f(1-x)}$

if $x \le 0$, $g(x) = 0$ ✓

$x \ge 1$, $g(x) = \dfrac{1}{1+0} = 1$ ✓

$x \in (0,1)$, $g(x) = \dfrac{[0,1)}{[0,1) + (0,1]} \Rightarrow (0,1)$ ✓

The sum of infinitely diff. fcns is infinitely diff. b/c they don't have an effect on each other when differentiating. $\square$

Rudin §5

2)

(4) if

$$C_0 + \frac{C_1}{2} + \dots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

where $C_0, \dots, C_n \in \mathbb{R}$

Prove

$$C_0 + C_1 x + \dots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 & 1.

Pf: $\sum_{k=0}^{n} \frac{C_k}{k+1} = 0 \Rightarrow \sum_{k=0}^{n} C_k x^k = 0$ has some $(x-\lambda)$ s.t. $\lambda \in (0,1)$ & $f(\lambda) = 0$

Let $f(x) = \sum_{k=0}^{n} \frac{C_k}{k+1} x^{k+1}$

$f(0) = 0$, clearly

$f(1) = \sum \frac{C_k}{k+1} = 0$ //

So if $f(0) = 0$, $f(1) = 0$, we can apply Rolle's to f.

$f'(x_0) = \sum_{k=0}^{n} C_k x^k = 0$, $x_0 \in (0,1)$ by Rolle's

so b/c there's an x_0 s.t. $\sum_{k=0}^{n} C_k x^k = 0$, we know

there is a root of $C_0 + C_1 x + \dots + C_{n-1} x^{n-1} + C_n x^n$ in between 0 & 1. $\square$

3) (8) Suppose $f'(x) > 0$ in (a,b). Prove that f is strictly $\uparrow$ in (a,b), & let g be its inverse fcn. Prove that g is diff & that $g'(f(x)) = \dfrac{1}{f'(x)}$ $(a < x < b)$

MVT: $[f(b) - f(a)]\, g'(c) = [g(b) - g(a)] \cdot f'(c)$

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$\cancel{g'(x) = \dfrac{1}{f'(x)}}$. So $f'(x)$ is cont on (a,b) & $[a,b]$ is cpt, so $f'(x)$ is unif. cont. Thus $\forall \varepsilon > 0$, $\exists \delta > 0$ st $|x-y| < \delta$, $|f'(x) - f'(y)| < \varepsilon$.

Pick $x_1, x_2 \in [a,b]$ w/ $|x_1 - x_2| < \delta$, $|f'(x_1) - f'(x_2)| < \varepsilon$

By MVT, $\exists$ $c \in (x_1, x_2)$ st $|c - x_2| < \delta$, Thus $|f'(c) - f'(x_2)| < \varepsilon$. $\square$

4) (18) Suppose f is a real fcn on $[a,b]$, n is a positive int, & $f^{(n-1)}$ exists for every $t \in [a,b]$. Let α, β, & P be as in 5.15.

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta} \qquad \text{for } t \in [a,b],\ t \neq \beta$$

differentiate $f(t) - f(\beta) = (t-\beta)\,Q(t)$ $n-1$ times @ $t = \alpha$ & derive the following

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}\alpha}{(n-1)!}(\beta - \alpha)^n$$

Proof by induction:

Base case, $n = 1$: $f'(t) = (t-\beta)\,Q(t) + f(\beta)$

$f'(t) = (t - \beta)\,Q'(t) + Q(t)$ ✓

Assume holds for $n-1$ cases: $f^{(n-1)}(t) = (n-1)\,Q^{(n-2)}(t) + (t-\beta)\,Q^{(n-1)}(t)$

take one more derivative:

$f^{(n)}(t) = n\,Q^{n-1}(t) + (t-\beta)\,Q^{(n)}(t)$

$\Rightarrow$ over

$$P(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k = f(\alpha) + \sum_{k=1}^{n-1} \frac{f^{(k)}\alpha}{k!}(\beta-\alpha)^k$$

$$= f(\alpha) + \sum_{k=1}^{n-1} \frac{k\, Q^{(k-1)}(\alpha) + (\alpha-\beta)\, Q^{(k)}(\alpha)}{k!}(\beta-\alpha)^k$$

$$= f(\alpha) + \sum \frac{Q^{(k-1)}(\alpha)}{(k-1)!}(\beta-\alpha)^k - \sum \frac{Q^{(k)}(\alpha)}{k!}(\beta-\alpha)^{k+1}$$

$$= f(\alpha) + \frac{Q(\alpha)}{0!}(\beta-\alpha) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n$$

$$= \cancel{f(\alpha)} + \left(f(\beta) - \cancel{f(\alpha)}\right) - \frac{Q^{(n-1)}\alpha}{(n-1)!}(\beta-\alpha)^n$$

$$P(\beta) = f(\beta) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n$$

$$\Rightarrow f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n \quad \checkmark \quad \square$$

5) (22) f is a real fcn on $(-\infty, \infty)$. Call x a fixed pt if $f(x)=x$

(a) If f is diff. & $f'(t) \neq 1$ for every real t, prove that f has at most 1 fixed point.

Assume f has 2 fixed pts x_1, x_2 st $f(x_1) = x_1$
$f(x_2) = x_2$

MVT says $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{x_2 - x_1}{x_2 - x_1} = 1 \rightarrow\leftarrow \square$

(b) $f(t) = t + (1+e^t)^{-1}$
for $f(t)=t$, $(1+e^t)^{-1} = 0$ $\quad \frac{1}{1+e^t} \neq 0 \;\; \forall t$, num $\neq 0$ $\square$

(c) $A<1$ st $|f'(t)| \leq A$ for all t, prove that a fixed pt x of f exists, & that $x = \lim x_n$, where x_1 is an arbitrary real number & $x_{n+1} = f(x_n)$, $n = 1, 2, \dots$

Assume for contradiction that $f(x) \neq x \;\; \forall x \in \mathbb{R}$

$\longrightarrow$ next

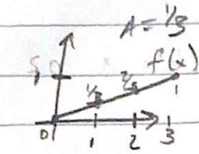
can partition all x's into 2 groups $f(a) < a$, $a \in \mathbb{R}$
$f(b) > b$, $b \in \mathbb{R}$

Pick 2 elements from the sets a_1, b_1

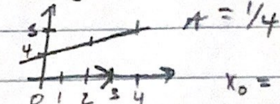
$\exists c \in (a_1, b_1)$ st $f'(c) = \dfrac{f(b_1) - f(a_1)}{b_1 - a_1} > 1$ $\rightarrow\leftarrow$ $\square$

$(f(b_1) > b_1,\ f(a_1) > a_1 \Rightarrow f'(c) > 1.)$

$x = \lim x_n$,



so x_n is increasing at a constant rate < 1.



$x_0 = 4$
$x_1 = 4.25$
$x_2 = 4.5$
$x_3 = 4.75$
$x_4 = 5$

$x_n = x_1 + \sum_{k=1}^{n} A \cdot k$

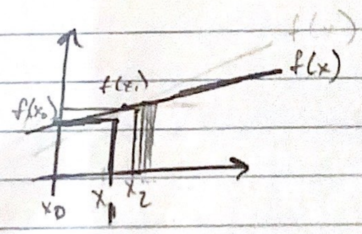
$|x_n - x| = |f(x_{n-1}) - f(x)|$

MVT: $\exists y \in (x_{n-1}, x)$ st $f'(y) = \dfrac{f(x_{n-1}) - f(x)}{x_{n-1} - x}$

$|f(x_{n-1}) - f(x)| = f'(y)\,|x_{n-1} - x| \leq A|x_{n-1} - x|$
$= |f(x_{n-2}) - f(x)|$
$\leq A^{n-1}|x_1 - x|$

$n \to \infty$, $A^{n-1}|x_1 - x| \to 0 \Rightarrow x = \lim x_n$ $\square$

(d) $(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to \ldots$



see graph, $d(x_n, x_{n-1})$ gets smaller as n increases.