HW 9

1. Since $e^{-\frac{1}{x}}$ is smooth, define
$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$
$f'(x)$ is also smooth.
construct a function $g(x) = \dfrac{f(x)}{f(x) + f(1-x)}$
~~Since~~ $f(1-x)$, is also $h(x) = \begin{cases} e^{-\frac{1}{1-x}} & \text{if } x < 1 \\ 0 & \text{if } x \ge 1. \end{cases}$
$\therefore g(x)$ is thus smooth and satisfies the constraints.

~~5.18~~

5.4) If we integrate we get
$f(x) = C_0 x + \frac{C_1}{2} x^2 + \dots \frac{C_n}{n+1} x^{n+1}$
$f(0) = 0$, $f(1) = 0$ as is given. By Rolle's theorem, there exists $c \in (0,1)$ s.t
$f'(x) = g(x) = 0$.

5.8) $|x - t| < \delta \Rightarrow$
$|f'(x) - f'(t)| < \varepsilon$
$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right|$
let $c \in (t, x)$. $= |f'(c) - f'(x)| < \varepsilon$
So uniformly diff.

5.18) By induction:
$f'(t) = (t - \beta) Q'(t) + Q(t) \Rightarrow Q(\alpha) = f'(\alpha) + Q'(\alpha)(\beta - \alpha)$
$\vdots$
$f^n(t) = (t - \beta) Q^n(t) + n Q^{n-1}(t) \Rightarrow Q^{n-1}(\alpha) = \frac{1}{n}(f^n(\alpha) + Q^n(\alpha)(\beta - \alpha))$
Repeat the same inductive process with $f(\beta) = f(\alpha) + (\beta - \alpha) Q(\alpha)$
$\vdots$
$P(\beta) + \frac{Q^{n-1}(\alpha)}{(n-1)!} (\beta - \alpha)^n$.

D as desired

5.22) a) Assume f has two fixed points, a and b, $f(a)=a$, $f(b)=b$.

then $\frac{f(b)-f(a)}{b-a} = \frac{b-a}{b-a} = 1$

So there must be $c \in (a, b)$ s.t. $f'(c) = 1$.

Contradiction.

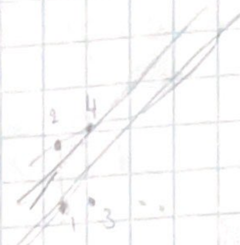
b) Assume a is a fixed point.

$f(a) = a + (1+e^{a})^{-1} = a$

$\frac{1}{1+e^{a}} = 0$

$\Rightarrow 1+e^{a} \to \infty$

which is impossible as $0 < f'(t) < 1 \ \forall t$.



c) $\cancel{(x_n, x_{n+1})}$ $\frac{x_n}{x_{n+1}} = 1$ $\frac{x_{n+1}-x_n}{x_n - x_{n-1}} = 0$

$\frac{f(x)}{f(f(x))}$ Since $|f'(x)| \leq A < 1 \ \forall x \in \mathbb{R}$

$\frac{f(f(x)) - f(x)}{f(x) - x} \leq A < 1$, $= f'(c)$

$f(f(x)) - f(x) < f(x) - x$

$\cancel{f(f(x)) = f(x)}$ $f(f(x)) < 2f(x) - x$

$x_{n+1} = f(x_n)$ $f(x_1) - x_1$

$x_2 - x$ WTS $\exists x$ s.t. $f(x) = x$

(x_1, x_2) (x_3, x_4)

$\frac{x_4 - x_2}{x_3 - x_1} = f'(x) < A$ $x \in (x_1, x_3)$

$f(x_2) - x_1 \leq A(f(y_1) - x_1)$

$f(x) - x = x_3 - x_1$

$f(f(x)) - x$

$f(x) = f(f(x))$ therefor $\exists$ fixed point.

d) The path describes the process in (c) because the sequence takes as input the previous output. When plotted, this is equivalent to recursively travelling to $f(x)=x$ every other point in the sequence.