HW 1 Math 104

$nr \geq 1$ $\quad n \geq \frac{1}{r}$ $\quad r > \frac{1}{n}$

4.14. a) $\sup(A+B) \geq a + b$

$\sup(A+B) - b \geq a \geq \sup A$ $\quad B$

$\sup(A+B) - \sup A \geq b \geq \sup B$

$\therefore \sup(A+B) \geq \sup A + \sup B$

$\sup(A) + \sup(B) \geq a + b \geq \sup(A+B)$

$\therefore \sup(A+B) = \sup A + \sup B$

b) $\inf(S) = -\sup(-S)$

This is intuitively true as we invert all $s \in S$.

Therefor, $\inf(A+B) = -\sup(-A-B)$

We know from part (a) that

$-\sup(-A-B) = -\sup(-A) - \sup(-B)$

$\therefore \boxed{\inf(A+B) = \inf(A) + \inf(B)}$

4.11. Suppose for contradiction the set of rationals, S, has size $|S| = k$. However, we can create another rational, $r \notin S$ by taking the largest rational number in S and using Denseness of $\mathbb{Q}$ to find a rational number between that and b. However, this is a contradiction, and so $|S|$ must be infinite.

1.10 $\sum_{x=0}^{k} 2n+2x+1)$ $\quad 4n-1 = 2n+2k+1$

$2n-2 \quad k = n-1$

$\sum_{x=0}^{n-1} 2n+2x+1$

$= 2n^2 + n \quad) + 2\left(\frac{(n-1)n}{2}\right)$

$= 2n^2 + n + n^2 - n$

$= \boxed{3n^2}$

1.12 a) This holds directly for 0 and 1

$n=2: (a+b)^2 = a^2 + 2ba + b^2 = \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2$

$n=3: (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

b) $\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{(n+1)!}{k!(n-k+1)!}$

$\frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{k} + \frac{1}{n-k+1}\right)$

$\hookrightarrow \left(\frac{n+1}{k(n-k+1)}\right)$

$= \boxed{\frac{(n+1)!}{k!(n-k+1)!}}$

c) $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$(a+b)^{n+1} = (a+b)^n (a+b)$

$= \left(\sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}\right)(a+b)$

$= \sum_{i=0}^{n} \binom{n}{i} a^{i+1} b^{n-i} + \sum \binom{n}{i} a^i b^{n-i+1}$

$= a^{n+1} + \sum_{i=1}^{n} \left(\binom{n}{i-1} + \binom{n}{i}\right) a^i b^{n-i+1} + b^{n+1}$

$= \sum_{i=0}^{n+1} \binom{n+1}{i} a^i b^{n-i+1}$

$\therefore$ True for $n+1$, by induction true for $n \in \mathbb{N}$.

2.1. $x^2 - 3 = 0$ By the rational root theorem, $p = \pm 1$, $q = \pm 3, \pm 1$ $x = \pm \frac{3}{1}, \pm 1$. None of these work so $\sqrt{3}$ must be irrational.

This same reasoning works for $\sqrt{5}$, $\sqrt{7}$, $\sqrt{24}$ and $\sqrt{31}$.

2.2 The same reasoning for 2.1 can be used here.

2.7. a) $\sqrt{4+\sqrt{8}}$

$= \sqrt{(1+\sqrt{3})^2} - \sqrt{3} = \boxed{1}$

b) $\sqrt{2+4+4\sqrt{2}} = \sqrt{(2+\sqrt{2})^2} - \sqrt{2}$

$= \boxed{2}$

3.6 a) $|a+(b+c)| \leq |a| + |b+c| \leq |a| + |b| + |c|$

b) basis $n=2$: $|a_1 + a_2| \leq |a_1| + |a_2|$

induct $n=k$ & $|(a_1 + \ldots a_k) + a_{k+1}| \leq |a_1 + \ldots a_k| + |a_{k+1}|$

we know $|a_1 + \ldots a_k| < |a_1| + \ldots |a_k|$

So true for $n \in \mathbb{N}$

7.5 a) $\sqrt{n^2+1} - n \frac{(\sqrt{n^2+1} + n)}{\sqrt{n^2+1} + n)} = \frac{1}{\sqrt{n^2+1} + n} = \boxed{0}$

b) Same as before $= \frac{n}{\sqrt{n^2+n} + n} = \frac{1}{\sqrt{1+\frac{1}{n}} + 1} = \boxed{\frac{1}{2}}$

c) $= \frac{4n^2+n-4n^2}{\sqrt{4n^2+n} + 2n} = \frac{n}{\sqrt{4n^2+n} + 2n} = \frac{1}{\sqrt{4+\frac{1}{n}} + 2} = \boxed{\frac{1}{4}}$