HW 2

10.7) $\forall \varepsilon > 0$, $\exists s \in S$ such that $\sup S - \varepsilon < s$.
Define $\varepsilon = \frac{1}{n}$ and construct a sequence
$S_n = \sup S - \frac{1}{n}$. $\lim_{n\to\infty} S_n = \sup S$.
$\forall x \in S_n$, $x \in S$ because $x = \sup S - \frac{1}{k}$, $k \geq 1$.

10.8) Let $S_{n+1} = S_n + K_n$ where $K_n = S_{n+1} - S_n$

$$\sigma_n = \frac{1}{n}\left(S_1 + (S_1 + K_1) + (S_1 + K_1 + K_2) \dots\right)$$

$$S_n = S_1 + \sum K_i$$

$$\sigma_n = \frac{1}{n}\left(S_1 \cdot n + \sum_{i=1}^{n} (n-i) K_i\right)$$

$$\sigma_{n+1} = \frac{1}{n+1}\left(S_1 \cdot n+1 + \sum_{i=1}^{n} (n-i) K_i\right)$$

$$\sigma_{n+1} - \sigma_n = \sum_{i=1}^{n+1} (n-i+1) K_i - \sum_{i=1}^{n} (n-i) K_i$$

$$= \sum_{i=1}^{n} (n-i) K_i + n - (n+1) - \sum_{i=1}^{n} (n-i) K_i$$

$$= K_{n+1} > 0$$

so σ is monotonic increasing.

<u>Squeeze Thm:</u>

$b_n \geq a_n$, $b_n \leq c_n$

$|a_n - L| < \varepsilon \quad \forall \varepsilon > 0,\ \forall n > N,\ N \geq 0$

$L - \varepsilon < a_n < \varepsilon + L$

$|c_n - L| < \varepsilon$

$L - \varepsilon < c_n < \varepsilon + L$

$L - \varepsilon < a_n \leq b_n \leq c_n < \varepsilon + L$

$-\varepsilon < b_n - L < \varepsilon$

$|b_n - L| < \varepsilon$, $\therefore \boxed{\lim_n b_n = L}$

9.9. a) Define $s_n' = s_{n+N_0}$, $n > 0$ $\lim_n s_n' = \lim_n s$

$t_n' = t_{n+N_0}$, $n > 0$.

$s_n' \leq t_n'$, so $\lim_n s_n' \leq \lim_n t_n'$
for all $n > 0$

Since $\lim_n s_n' = +\infty$, t_n' must be $+\infty$, and so

b) " similar reasoning as (a),
Since $\lim s_n' \leq \lim t_n' = -\infty$,
$\lim s_n' = \boxed{-\infty}$

c) let $\lim s_n = S$, $\lim t_n = T$, $\exists N \ \forall \varepsilon > 0$ s.t $\forall n > N$,
$|s_n - S| < \varepsilon$, $|t_n - T| < \varepsilon$
Since $s_n \leq t_n$, $\boxed{S \leq T}$ is implied

9.15. $\frac{1}{n!} \le \frac{1}{(\frac{n}{2})^n}$

$\frac{a^n}{n!} \le \frac{a^n}{(\frac{n}{2})^n} = \left(\frac{a}{\frac{n}{2}}\right)^n = 0$ as $n \to \infty$

So $\boxed{\frac{a^n}{n!} \Rightarrow 0}$

10.9 a) $S_2 = \frac{1 \cdot S_1^2}{2} = \frac{1}{2}$ $S_3 = \frac{2}{3} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{6}$

$S_4 = \frac{3}{4} \cdot \left(\frac{1}{6}\right)^2 = \frac{1}{48}$

b) S_n is monotonic and decreasing. $S_n > 0$.

Pf $S_n = \frac{n}{n+1} S_{n-1}^2$

$\frac{n}{n+1} S_{n-1}^2 - S_{n-1}$

$= S_{n-1} \left(\frac{n}{n+1} S_{n-1} - 1\right)$

$S_{n-1} \le 1$ since $S_n = \frac{n}{n+1} S_{n-1}^2$

so is negative.

$\therefore$ lim exists

c) $\lim_n \frac{n}{n+1} \lim S_n^2 = 0$. $\lim S_n^2 = 0$

10.10 a) $S_2 = \frac{2}{3}$, $S_3 = \frac{5}{9}$, $S_4 = \frac{14}{27}$

b) $S_1 > \frac{1}{2}$; $S_{n+1} = \frac{1}{3}(S_n + 1)$ $\frac{1}{3} S_n + \frac{1}{3} > \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3} = \frac{1}{2}$

c) $S_n > \frac{1}{2}$ $S_{n+1} = \frac{1}{3}(S_n + 1) < \frac{1}{3} S_n + \frac{1}{3} = \frac{2}{3} S_n = S_n$

d) $3S = S + 1$, so $2S = 1$. ? ?

10.11 a) $t_{n+1} < t_n$ since $1 - \frac{1}{4n^2} < 1$ $\forall n$

b) 0