HW 8

1. $\forall \varepsilon > 0$, $\lim_{n\to\infty} \sup\{|\frac{n+\sin x}{2n+\cos n^2 x} - \frac{1}{2}|, \forall x \in \mathbb{R}\} = 0$

$|n(\frac{1+\sin x/n}{2+\cos n^2 x/n}) - \frac{1}{2}| < \varepsilon + \frac{1}{2}$

$n + \sin x \le n+1$

$\frac{n+\sin x}{2n+\cos n^2 x} \le \frac{n+1}{2n+\cos n^2 x} \le \frac{n+1}{2n-1}$

$\frac{n+1}{2n-1} - \frac{1}{2} = \varepsilon$ $\quad \frac{N+1}{2N-1} = \frac{3\varepsilon}{2}$

$2N+2 = 6N\varepsilon - 3\varepsilon$

$2N(1-3\varepsilon) = -2-3\varepsilon$

$\frac{3\varepsilon+2}{3\varepsilon-1}$ $\quad N = \boxed{\frac{3\varepsilon+2}{3\varepsilon-1}}$

$\therefore$ For all $n > N$, we have $\lim_{n\to\infty} f_n = f = \frac{1}{2}$, for all $\varepsilon > 0$, for all $x \in \mathbb{R}$. $\square$

2. If $\sum_n |a_n| < \infty$, $\sum_n |a_n| \le b$, $b < \infty$

$\therefore f(x) = \sum_{n=1}^{\infty} a_n x^n < \sum b x^n < b \sum_{n=1}^{\infty} x^n$

~~$\forall \varepsilon > 0$, $|x+p| < \varepsilon$ $\quad |f(x) - f(p)| < \delta$~~

~~$f(x) - f(p) < b \sum x^n - b \sum p^n$~~

$f_n(x) \le \sum a_n$.

Since $\sum |a_n|$ converges, ~~and $|a_n|$ is continuous~~ $|a_n| \to a$, $f_n \le a_n$ so f_n is also convergent, so $f_n \to f$. Since each f_n is continuous, and uniform convergence preserves continuity, f is continuous.

By the same logic, $\sum \frac{x^n}{n^2} \le \sum \frac{1}{n^2}$ which converges, so $\sum x^n/n^2$ is continuous.

3. Let $a \in (0,1)$. $|f(x)| \le a^n$ $\forall n \ge 1$ $\forall x \in [-a, a]$

$\sum a^n$ converges ~~by~~ since $a < 1$, so $f(x)$ is ~~uniformly convergent~~ to 0 on ~~(-1, 1)~~ uniform convergent on $[-a, a]$.

Since $a \in (0,1)$, and each f_n is continuous, f is continuous on $(-1, 1)$.

This convergence is not uniform on $(-1,1)$, however, because let $a_n = \frac{n}{n+1}$

$a_n \to 1$, $a_n \in (-1, 1)$, $f(a_n)$ does not converge to $f(1)$.