9.9 Suppose $\exists N_0$ such that $s_n \le t_n$ for $\forall n > N_0$

(a) Let $\lim s_n = +\infty$.

By Definition 9.8, for each $M>0$, there is a number N_1 such that $n > N_1$ implies $s_n > M$.

Let $N = \max\{N_0, N_1\}$.

Then $n > N$ implies $M < s_n \le t_n$

Thus, for each $M>0$, $n>N$ implies $t_n > M$.

Therefore, $\lim t_n = +\infty$

(b) Let $\lim t_n = -\infty$.

By Definition 9.8, for each $M<0$, there is a number N_1 such that $n>N_1$ implies $t_n < M$.

Let $N = \max\{N_0, N_1\}$

Then $n > N$ implies $s_n \le t_n < M$

Thus, for each $M<0$, $n>N$ implies $s_n < M$.

Therefore, $\lim s_n = -\infty$

(c) Let $\lim s_n = s$ $\lim t_n = t$, need to show $s \le t$.

Then $\exists N_1 \in \mathbb{N}$, such that $n > N_1$ implies $|s_n - s| < \varepsilon$

$\exists N_2 \in \mathbb{N}$, such that $n > N_2$ implies $|t_n - t| < \varepsilon$

$s - \varepsilon < s_n < s + \varepsilon$ $\quad$ $t - \varepsilon < t_n < t + \varepsilon$

Let $N = \max\{N_0, N_1, N_2\}$, $n > N$ implies

$|s_n - s| < \varepsilon$ $\quad$ $|t_n - t| < \varepsilon$ $\quad$ $s_n \le t_n$

Suppose $s > t$ (proof by contradiction)

Let $\varepsilon = \frac{s-t}{2}$

$\Rightarrow \frac{s+t}{2} < s_n < \frac{3s-t}{2}$ $\quad$ $\frac{3t-s}{2} < t_n < \frac{s+t}{2}$

$\Rightarrow s_n > t_n$ which contradicts with $s_n \le t_n \ \forall n > N_0$

Therefore, $s \le t$, that is $\lim s_n \le \lim t_n$.

9.15 Show $\lim_{n\to\infty} \frac{a^n}{n!} = 0 \ \forall a \in \mathbb{R}$

Let $s_n = \frac{a^n}{n!}$

$$\frac{s_{n+1}}{s_n} = \frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n} = \frac{a}{n+1}$$

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \lim_{n\to\infty} \frac{a}{n+1} = a \lim_{n\to\infty} \frac{1}{n+1} = a \cdot 0 = 0 < 1$$

By Archimedian Principle, $\exists k$ such that $0 < k < 1$

Since $\lim_{n\to\infty} \frac{s_{n+1}}{s_n}$ exists, then $\exists N \in \mathbb{N}$ such that $n > N$ implies $\frac{s_{n+1}}{s_n} < k$

So $s_{n+1} < k s_n$ for $n > N$

$$s_{N+1} < k s_N \qquad s_{N+2} < k s_{N+1} < k \cdot k s_n = k^2 s_n$$

$$\Rightarrow s_n < k^{n-N} s_N$$

Since $0 < k < 1$, $k^{n-N} \to 0$.

Thus, $\lim_{n\to\infty} s_n = 0$, that is $\lim_{n\to\infty} \frac{a^n}{n!} = 0$.

10.7 By definition of supremum, $\forall \varepsilon > 0$, $\exists s \in S$ such that $\sup S - s < \varepsilon$,

which implies $s_n > \sup S - \varepsilon$.

Thus $\sup S - \varepsilon$ is not an upper bound of s_n.

Take $\varepsilon = \frac{1}{n}$.

$\exists s_n \in S$ such that $\sup S - \frac{1}{n} < s_n < \sup S$

$$\lim_{n\to\infty} (\sup S - \tfrac{1}{n}) = \lim_{n\to\infty} \sup S - \lim_{n\to\infty} \tfrac{1}{n} = \sup S - 0 = \sup S$$

$$\lim_{n\to\infty} (\sup S) = \sup S$$

By squeeze theorem, $\lim s_n = \sup S$.

10.8 To prove (σ_n) is an increasing sequence, need to show $\sigma_n \le \sigma_{n+1} \ \forall n$.

Since (s_n) is an increasing sequence of positive numbers, $s_n \le s_{n+1}$ $\forall n$.

$$\begin{aligned}
\sigma_{n+1} - \sigma_n &= \frac{1}{n+1}(s_1+s_2+\cdots+s_{n+1}) - \frac{1}{n}(s_1+s_2+\cdots+s_n) \\
&= \frac{1}{n(n+1)}\left[n(s_1+s_2+\cdots+s_{n+1}) - (n+1)(s_1+s_2+\cdots+s_n)\right] \\
&= \frac{1}{n(n+1)}\left[n s_{n+1} - (s_1+s_2+\cdots+s_n)\right] \\
&= \frac{1}{n(n+1)}\left[(s_{n+1}-s_1)+(s_{n+1}-s_2)+\cdots+(s_{n+1}-s_n)\right] \\
&\geqslant 0
\end{aligned}$$

Thus $\sigma_{n+1} \geqslant \sigma_n$ $\forall n$.

Therefore, (σ_n) is an increasing sequence.

10.9 (a) $s_2 = \frac{1}{1+1}s_1^2 = \frac{1}{2}$

$s_3 = \frac{2}{2+1}s_2^2 = \frac{1}{6}$

$s_4 = \frac{3}{3+1}s_3^2 = \frac{1}{48}$

(b) By Corollary 10.5, if (s_n) is a monotone sequence, then $\lim s_n$ is always meaningful.

Claim: (s_n) is a monotonic decreasing sequence.

Proof by induction.

Basis: $n=1$ $\quad s_1 > s_2$

Inductive step: Assume $s_n > s_{n+1}$ holds, remains to show $s_{n+1} > s_{n+2}$

$$\begin{aligned}
s_{n+1} - s_{n+2} &= \frac{n}{n+1}s_n^2 - \frac{n+1}{n+2}s_{n+1}^2 \\
&= \frac{n}{n+1}s_n^2 - \frac{n+1}{n+2}\left(\frac{n}{n+1}s_n^2\right)^2 \\
&= \frac{n}{n+1}s_n^2 - \frac{n^2}{(n+1)(n+2)}s_n^4 \\
&= \frac{n}{n+1}s_n^2\left(1 - \frac{n}{n+2}s_n^2\right) \\
&> 0
\end{aligned}$$

$0 < s_n < s_1 = 1$

$\Rightarrow 0 < s_n^2 < 1 \quad \frac{n}{n+2} < 1$

$\Rightarrow \frac{n}{n+2}s_n^2 < 1$

$\Rightarrow s_{n+1} > s_{n+2}$

By principle of mathematical induction, $s_n > s_{n+1}$ $\forall n$.

Thus, (s_n) is a monotonic decreasing sequence.

(s_n) is also bounded $0 < s_n \le 1$.

By Theorem 10.2, (s_n) is convergent, thus $\lim s_n$ exists.

(c) Let $\lim s_n = s$.

$$s = \lim s_{n+1} = \lim \frac{n}{n+1} s_n^2 = \lim \frac{n}{n+1} \cdot \lim s_n^2$$

$$= 1 \cdot \lim s_n^2 = s^2$$

$\Rightarrow s = 1$ or $s = 0$

Since (s_n) is monotonic decreasing and $s_2 < 1$, it is impossible to have $\lim s_n = 1$.

Therefore $\lim s_n = 0$.

10.10 (a) $s_2 = \frac{1}{3}(s_1 + 1) = \frac{2}{3}$

$s_3 = \frac{1}{3}(s_2 + 1) = \frac{5}{9}$

$s_4 = \frac{1}{3}(s_3 + 1) = \frac{14}{27}$

(b) Basis step: $n = 1 \quad s_1 = 1 > \frac{1}{2}$

Inductive step: Assume $s_n > \frac{1}{2}$, need to show $s_{n+1} > \frac{1}{2}$.

$$s_{n+1} = \frac{1}{3}(s_n + 1) = \frac{1}{3}s_n + \frac{1}{3}$$

$$> \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3}$$

$$= \frac{1}{2}$$

Thus, $s_{n+1} > \frac{1}{2}$ if $s_n > \frac{1}{2}$.

By principle of mathematical induction, $s_n > \frac{1}{2} \ \forall n$.

(c) $$s_n - s_{n+1} = s_n - \frac{1}{3}(s_n + 1) = s_n - \frac{1}{3}s_n - \frac{1}{3}$$

$$= \frac{2}{3}s_n - \frac{1}{3}$$

$$> \frac{2}{3} \cdot \frac{1}{2} - \frac{1}{3} = 0$$

$\Rightarrow s_n > s_{n+1} \ \forall n$

Therefore, (s_n) is a decreasing sequence.

(d) By part (b) (c), (s_n) is monotonic decreasing and bounded sequence, thus (s_n) is convergent and $\lim s_n$ exists.

Let $\lim s_n = s$.

$$s = \lim s_{n+1} = \lim\left(\frac{1}{3}(s_n+1)\right) = \frac{1}{3}\lim(s_n+1)$$

$$= \frac{1}{3}(\lim s_n + 1) = \frac{1}{3}(s+1)$$

$$\Rightarrow s = \frac{1}{2}$$

Thus, $\lim s_n = \frac{1}{2}$.

10.11 (a) $t_1 = 1 \quad t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n$

Since $0 < \frac{1}{4n^2} < 1$, then $0 < 1 - \frac{1}{4n^2} < 1$.

$\Rightarrow t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n < t_n$.

Thus, (t_n) is a monotonic decreasing sequence.

Since $t_1 = 1$, $0 < 1 - \frac{1}{4n^2} < 1$, then $0 < t_n \leq 1$.

Thus, (t_n) is bounded.

By Theorem 10.2, (t_n) is convergent, so $\lim t_n$ exists.

(b) $0 < \lim t_n < 1$

$t_1 = 1 \quad t_2 = \frac{3}{4} = 0.75 \quad t_3 = \frac{45}{64} = 0.7031 \quad t_4 = \frac{35 \cdot 45}{36 \cdot 64} = 0.6836$

$t_5 = \frac{63 \cdot 35 \cdot 45}{64 \cdot 36 \cdot 64} = 0.6729 \quad \cdots$

$\lim t_n = 0.6$?

<u>Squeeze test</u>. Let a_n, b_n, c_n be three sequences, such that $a_n \le b_n \le c_n$, and $L = \lim a_n = \lim c_n$. Show that $\lim b_n = L$.

<u>Proof</u>: Since $L = \lim a_n = \lim c_n$.

$\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ such that $n > N_1$ implies $|a_n - L| < \varepsilon$

$\exists N_2 \in \mathbb{N}$ such that $n > N_2$ implies $|c_n - L| < \varepsilon$

Let $N = \max\{N_1, N_2\}$

$n > N$ implies $L - \varepsilon < a_n < L + \varepsilon$ $\quad L - \varepsilon < c_n < L + \varepsilon$

Since $a_n \le b_n \le c_n$ whenever $n > N$,

then $b_n \geq a_n > L - \varepsilon$, $b_n \le c_n < L + \varepsilon$, so $L - \varepsilon < b_n < L + \varepsilon$.

$\Rightarrow |b_n - L| < \varepsilon \quad \forall n > N$

Therefore $\lim b_n = L$.