10.6 a) By Definition 10.8, a sequence (s_n) of real numbers is Cauchy sequence if for $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $m, n > N$ implies $|s_m - s_n| < \varepsilon$.

For all $n \in \mathbb{N}$, $|s_{n+1} - s_n| < 2^{-n}$

For all $m, n \in \mathbb{N}$. WLOG, assuming $m > n$, then:

$$
\begin{aligned}
|s_m - s_n| &\le |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \cdots + |s_{n+1} - s_n| \\
&< 2^{-(m-1)} + 2^{-(m-2)} + \cdots + 2^{-n} \\
&= 2^{-n}\left(2^0 + 2^{-1} + 2^{-2} + \cdots + 2^{-(m-n-1)}\right) \\
&< 2^{-n} \sum_{i=0}^{\infty} 2^{-i} \\
&= 2^{-(n-1)}
\end{aligned}
$$

Thus, $|s_m - s_n| < 2^{-(n-1)}$.

Let $t_n = 2^{-(n-1)}$, t_n converges to 0.

Hence, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N$ implies $|t_n - 0| < \varepsilon$,

that is, $t_n < \varepsilon$.

Thus, $|s_m - s_n| < 2^{-(n-1)} = t_n < \varepsilon$.

Therefore, (s_n) is a Cauchy sequence and hence a convergent sequence.

b) The result in (a) is not true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$. In part (a), we utilized the fact that $2^{-n} \sum_{i=0}^{\infty} 2^{-i} = 2^{-(n-1)}$ converges to 0. However, if $|s_{n+1} - s_n| < \frac{1}{n}$, then $|s_m - s_n| \leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \cdots + |s_{n+1} - s_n| < \frac{1}{m-1} + \frac{1}{m-2} + \cdots + \frac{1}{n}$.
$\sum_{n=0}^{m-1} \frac{1}{n}$ diverges to $+\infty$ when $m \to \infty$, so the distance between s_n and s_m can grow into infinity, thus (s_n) might not be Cauchy.

11.2 • $a_n = (-1)^n$

a) monotone subsequence: $a_{n_k} = (1, 1, 1, \cdots)$

The selection function is given by $\sigma(k) = 2k$

b) $S_a = \{1, -1\}$

c) $\limsup a_n = 1$ $\liminf a_n = -1$

- $b_n = \frac{1}{n}$

a) monotone subsequence : b_n itself is monotone decreasing

b) $S_b = \{0\}$

c) $\limsup b_n = 0$ $\liminf b_n = 0$

- $c_n = n^2$

a) monotone subsequence : c_n itself is monotone increasing

b) $S_c = \{+\infty\}$

c) $\limsup c_n = +\infty$ $\liminf c_n = +\infty$

- $d_n = \frac{6n+4}{7n-3}$

a) monotone subsequence : d_n itself is monotone decreasing

b) $S_d = \{\frac{6}{7}\}$

c) $\limsup d_n = \frac{6}{7}$ $\liminf d_n = \frac{6}{7}$

---

d) Sequence (a_n) diverges

Sequence (b_n) converges to 0

Sequence (c_n) diverges to $+\infty$

Sequence (d_n) converges to $\frac{6}{7}$

No sequence diverges to $-\infty$

e) Sequence (a_n) is bounded by $[-1, 1]$

Sequence (b_n) is bounded by $(0, 1]$

Sequence (c_n) is not bounded (bounded below by 1, not bounded above)

Sequence (d_n) is bounded by $(\frac{6}{7}, \frac{5}{2}]$

11.3 • $s_n = \cos(\frac{n\pi}{3})$

a) monotone subsequence : $s_{n_k} = (1, 1, 1, \cdots)$

The selection function is given by $\sigma(k) = 6k$

b) $S_s = \{-\frac{1}{2}, \frac{1}{2}, -1, 1\}$

c) $\limsup s_n = 1$ $\liminf s_n = -1$

• $t_n = \frac{3}{4n+1}$

a) monotone subsequence : t_n itself is monotone decreasing

b) $S_t = \{0\}$

c) $\limsup t_n = 0$ $\liminf t_n = 0$

• $u_n = (-\frac{1}{2})^n$

a) monotone subsequence : $u_{n_k} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$

The selection function is given by $\sigma(k) = 2k$.

b) $S_u = \{0\}$

c) $\limsup u_n = 0$ $\liminf u_n = 0$

• $v_n = (-1)^n + \frac{1}{n}$

a) monotone subsequence : $v_{n_k} = (1+\frac{1}{2}, 1+\frac{1}{4}, 1+\frac{1}{8}, \cdots)$

The selection function is given by $\sigma(k) = 2k$.

b) $S_v = \{-1, 1\}$

c) $\limsup v_n = 1$ $\liminf v_n = -1$

---

d) (s_n) diverges

(t_n) converges to 0

(u_n) converges to 0

(v_n) diverges

e) (s_n) is bounded by $[-1,1]$

(t_n) is bounded by $(0, \frac{3}{5}]$

(u_n) is bounded by $[-\frac{1}{2}, \frac{1}{4}]$

(v_n) is bounded by $(-1, \frac{3}{2}]$

11.5 a) The set of subsequential limits for (q_n) is $[0,1]$.

By Theorem 11.2, if t is in $\mathbb{R}$, then there is a subsequence of (s_n) converging to t if and only if the set $\{n \in \mathbb{N} : |s_n - t| < \varepsilon\}$ is infinite for all $\varepsilon > 0$.

Consider $\forall t \in (0,1]$ $t \in \mathbb{R}$, $\forall \varepsilon > 0$.

By denseness of $\mathbb{Q}$, the number of rationals in the interval $(t-\varepsilon, t+\varepsilon)$ is infinite, thus $\{n \in \mathbb{N} : |q_n - t| < \varepsilon\}$ is infinite.

Therefore, all elements of $(0,1]$ are subsequential limits.

Since $(0, 0+\varepsilon)$ contains infinite number of rationals, $\{n \in \mathbb{N} : |q_n - 0| < \varepsilon\}$ is infinite, 0 is also a subsequential limit.

Therefore the set of subsequential limit is $[0,1]$.

b) $\limsup q_n = \sup S = \sup [0,1] = 1$

$\liminf q_n = \inf S = \inf [0,1] = 0$

- What is lim sup?

Definition: Let (s_n) be a sequence in $\mathbb{R}$.

$\limsup s_n = \lim_{N \to \infty} \sup\{s_n : n > N\}$

if (s_n) is not bounded above, $\limsup s_n = +\infty$.

$\limsup s_n \leq \sup\{s_n : n \in \mathbb{N}\}$

Difference between lim sup and sup:

sup: least upper bound of the sequence (s_n) as a set

input: set

may or may not contained in the set

lim sup: limit of the sequence (a_n), where $a_n = \sup_{k \geq n} s_k$

input: sequence

lim sup always exists