Ross

12.10 Prove (s_n) is bounded if and only if $\limsup |s_n| < +\infty$

$\Rightarrow$ If (s_n) is bounded, then $\exists M \in \mathbb{R}$ such that $|s_n| \le M$ for all n.

Therefore, $\sup\{|s_n| : n > N\} \le M < +\infty$ for all n

Hence, $\limsup |s_n| = \lim_{N\to\infty} \sup\{|s_n| : n > N\} < +\infty$

$\Leftarrow$ If $\limsup |s_n| < +\infty$, then let $M = \limsup |s_n|$, thus

$\sup\{|s_n| : n > N\}$ converges to M.

For $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|\sup\{|s_n| : n > N\} - M| < \varepsilon$

$\Rightarrow \sup\{|s_n| : n > N\} < M + \varepsilon$

$\Rightarrow |s_n| < M + \varepsilon$ for all $n > N$

Construct $U = \max\{|s_1|, |s_2|, \cdots, |s_N|, M+\varepsilon\}$

Thus, for all n, $|s_n| \le U$

Therefore, (s_n) is bounded.

12.12 (s_n) is a seq_n of nonnegative numbers

$\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$

a) $\liminf \sigma_n \le \limsup \sigma_n$ is obvious.

If $\limsup s_n = +\infty$, then $\limsup \sigma_n \le \limsup s_n$ holds.

For $\limsup s_n < +\infty$,

Let $M, N \in \mathbb{R}$, $M > N$

For $n > M$, $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n) = \frac{1}{n}(s_1 + s_2 + \cdots + s_N) + \frac{1}{n}(s_{N+1} + \cdots + s_n)$

$$\le \frac{1}{M}(s_1 + \cdots + s_N) + \frac{1}{n}(s_{N+1} + \cdots + s_n)$$

$$\le \frac{1}{M}(s_1 + \cdots + s_N) + \frac{n-N}{n}\sup\{s_n : n > N\}$$

$$\le \frac{1}{M}(s_1 + \cdots + s_N) + \sup\{s_n : n > N\}$$

Thus, $\sup\{\sigma_n : n > M\} \le \frac{1}{M}(s_1 + \cdots + s_N) + \sup\{s_n : n > N\}$

$$\limsup \sigma_n = \lim_{M\to\infty} \sup\{\sigma_n : n > M\} \le \lim_{M\to\infty} \left(\frac{1}{M}(s_1+\cdots+s_N)\right) + \lim_{N\to\infty} \sup\{s_n : n > N\}$$
$$\le \lim_{M\to\infty}\left(\frac{1}{M}\right)(s_1+\cdots+s_N) + \lim_{N\to\infty} \sup\{s_n : n > N\}$$
$$\le \lim_{N\to\infty} \sup\{s_n : n > N\}$$
$$= \limsup s_n$$

Since $\limsup s_n = -\liminf(-s_n)$, $\limsup \sigma_n = -\liminf(-\sigma_n)$,

$-\sigma_n = \frac{1}{n}(-s_1 - s_2 - \cdots - s_n)$, then $\limsup(-\sigma_n) \le \limsup(-s_n)$.

Thus, $-\limsup(-\sigma_n) \ge -\limsup(-s_n)$.

Hence, $\liminf s_n \le \liminf \sigma_n$

Therefore, $\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n$

b) If $\lim s_n$ exists, then by Theorem 10.7, $\liminf s_n = \lim s_n = \limsup s_n$.

By part (a), $\lim s_n = \liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n = \lim s_n$

Thus, $\liminf \sigma_n = \limsup \sigma_n = \lim s_n$.

By Theorem 10.7, $\lim \sigma_n$ exists and $\lim \sigma_n = \limsup \sigma_n = \liminf \sigma_n = \lim s_n$.

c) Example:

$s_n = (1, -1, 1, -1, \dots)$

$\lim s_n$ does not exist

$\lim \sigma_n = 0$

14.2 a) $\sum \frac{n-1}{n^2} = \sum\left(\frac{1}{n} - \frac{1}{n^2}\right) = \sum \frac{1}{n} - \sum \frac{1}{n^2}$

not converge, since $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges

b) $(-1)^n = \begin{cases} 0 & n \text{ is even} \\ -1 & n \text{ is odd} \end{cases}$ $\quad \sum_{n=1}^{2n} (-1)^n = 0 \quad \sum_{n=1}^{2n+1} (-1)^n = -1$

not converge, since $\sum (-1)^n$ is oscillating between 0 and -1

c) $\sum \frac{3n}{n^3} = 3\sum \frac{n}{n^3} = 3\sum \frac{1}{n^2}$

converge, since $\sum \frac{1}{n^2}$ converges

d) $\sum \frac{n^3}{3^n}$

Apply the Root Test, $\limsup |\frac{n^3}{3^n}|^{\frac{1}{n}} = \lim\sup |\frac{(n^{\frac{1}{n}})^3}{3}|$.

Since $\lim n^{\frac{1}{n}} = 1$, then $\lim (n^{\frac{1}{n}})^3 = 1$.

Thus $\lim\sup |\frac{(n^{\frac{1}{n}})^3}{3}| = \frac{1}{3} < 1$

By the Root Test, $\sum \frac{n^3}{3^n}$ converges.

e) $\sum \frac{n^2}{n!}$

Apply the Ratio Test, $\lim\sup |\frac{a_{n+1}}{a_n}| = \lim\sup |\frac{n+1}{n^2}| = \lim\sup |\frac{1}{n} + \frac{1}{n^2}| = 0 < 1$

By the Ratio Test, $\sum \frac{n^2}{n!}$ converges.

f) $\sum \frac{1}{n^n}$

Apply the Root Test, $\lim\sup |\frac{1}{n^n}|^{\frac{1}{n}} = \lim\sup |\frac{1}{n}| = 0 < 1$

By the Root Test, $\sum \frac{1}{n^n}$ converges.

g) $\sum \frac{n}{2^n}$

Apply the Root Test, $\lim\sup |\frac{n}{2^n}|^{\frac{1}{n}} = \lim\sup |\frac{n^{\frac{1}{n}}}{2}|$

Since $\lim n^{\frac{1}{n}} = 1$, then $\lim\sup |\frac{n^{\frac{1}{n}}}{2}| = \frac{1}{2} < 1$

By the Root Test, $\sum \frac{n}{2^n}$ converges.

14.10 $\sum a_n$ diverges by the Root Test, no information by the Ratio Test

$\lim\sup |a_n|^{\frac{1}{n}} > 1$ $\qquad$ $\lim\inf |\frac{a_{n+1}}{a_n}| \le 1 \le \lim\sup |\frac{a_{n+1}}{a_n}|$

$a_n = 2^{(-1)^n \cdot n}$

$\lim\sup |2^{(-1)^n \cdot n}|^{\frac{1}{n}} = \lim\sup |2^{(-1)^n}| = 2 > 1$

By the Root Test, $\sum a_n$ diverges.

$|\frac{a_{n+1}}{a_n}| = 2^{(-1)^{n+1}(2n+1)}$

$\lim\inf |\frac{a_{n+1}}{a_n}| = \lim 2^{-2n-1} = 0$

$\lim\sup |\frac{a_{n+1}}{a_n}| = \lim 2^{2n+1} = \infty$

By the Ratio Test gives no information about the series $\sum a_n$.

Rudin

6\. a. $a_n = \sqrt{n+1} - \sqrt{n}$

$$S_n = a_1 + a_2 + \cdots + a_n = (\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + \cdots + (\sqrt{n+1}-\sqrt{n})$$

$$= \sqrt{n+1} - \sqrt{1} = \sqrt{n+1} - 1$$

The partial sums diverge, so the series Σa_n diverges.

b. $a_n = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{n(\sqrt{n+1}+\sqrt{n})} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} < \frac{1}{n\cdot 2\sqrt{n}} = \frac{1}{n^{3/2}}$ $\quad \frac{3}{2} > 1$

By p-series test, the series Σa_n converges.

c. $a_n = (\sqrt[n]{n} - 1)^n$

Apply the Root Test,

$$\limsup |a_n|^{\frac{1}{n}} = \limsup |\sqrt[n]{n} - 1| = 0 < 1$$

By the Root Test, the series Σa_n converges.

d. $a_n = \frac{1}{1+z^n}$

If $|z| \le 1$, then $|a_n| \ge \frac{1}{2}$, $\lim a_n = 1 \ne 0$, the series Σa_n diverges.

If $|z| > 1$, then $|a_n| < \frac{1}{2}$, $a_n = \frac{1}{1+z^n} < \frac{1}{z^n}$

Apply the Root Test $\limsup |\frac{1}{z^n}|^{\frac{1}{n}} = \limsup |z^{-1}| = \frac{1}{z} < 1$

By the Root Test, the series $\Sigma \frac{1}{z^n}$ converges.

By Comparison Test, the series Σa_n converges.

7\. Since Σa_n converges, $\lim a_n = 0$ $\quad a_n < 1$ $n \to \infty$

Since $(\sqrt{a_n} - \frac{1}{n})^2 \ge 0$, $\frac{\sqrt{a_n}}{n} \le \frac{1}{2}(a_n^2 + \frac{1}{n^2})$

Since $a_n < 1$, $a_n^2 < a_n$, then by Comparison Test, Σa_n^2 converges.

Since $\Sigma \frac{1}{n^2}$ also converges, by Comparison Test, $\Sigma \frac{\sqrt{a_n}}{n}$ converges.

9\. a) $\limsup |n^3|^{\frac{1}{n}} = \limsup |n^{3/n}| = \limsup |(n^{\frac{1}{n}})^3| = 1$

radius of convergence is 1

b) Apply the Ratio Test,

$\limsup \left|\frac{a_{n+1}}{a_n}\right| = \limsup \left|\frac{2}{n+1}\right| = 0$

radius of convergence is ∞

c) Apply the Ratio Test,

$\limsup \left|\frac{a_{n+1}}{a_n}\right| = \limsup \left|2\cdot\left(\frac{n}{n+1}\right)^2\right| = 2$

radius of convergence is $\frac{1}{2}$

d) $\limsup \left|\frac{n^3}{3^n}\right|^{\frac{1}{n}} = \limsup \left|\frac{n^{3/n}}{3}\right| = \frac{1}{3}$

radius of convergence is 3

11. a) Σa_n diverges

If a_n is not bounded, then $\lim \frac{a_n}{1+a_n} = \lim\left(1-\frac{1}{1+a_n}\right) \neq 0$

If a_n is bounded, then $\exists M$ s.t. $a_n \le M$ $\forall n \in \mathbb{N}$, $\frac{a_n}{1+a_n} \geq \frac{a_n}{1+M}$.

Therefore, $\Sigma \frac{a_n}{1+a_n}$ diverges.

b) $\frac{a_{N+1}}{S_{N+1}} + \cdots + \frac{a_{N+k}}{S_{N+k}} \geq \frac{a_{N+1}}{S_{N+k}} + \cdots + \frac{a_{N+k}}{S_{N+k}}$

$= \frac{1}{S_{N+k}}(a_{N+1} + \cdots + a_{N+k}) = \frac{1}{S_{N+k}}(S_{N+k} - S_N)$

$= 1 - \frac{S_N}{S_{N+k}}$

$\Sigma \frac{a_n}{S_n}$ diverges since it does not a Cauchy sequence

c) $\frac{1}{S_{n-1}} - \frac{1}{S_n} = \frac{S_n - S_{n-1}}{S_{n-1}S_n} = \frac{a_n}{S_{n-1}S_n} \geq \frac{a_n}{S_n^2}$ since $S_{n-1} < S_n$

$\Sigma\left(\frac{1}{S_{n-1}} - \frac{1}{S_n}\right)$ converges (?)

By Comparison test, $\Sigma \frac{a_n}{S_n^2}$ converges.

d) $\Sigma \frac{a_n}{1+na_n}$ convergent or divergent

$\Sigma \frac{a_n}{1+n^2 a_n}$ convergent