Ross

13.3 (a) $d(x,y) = \sup\{|x_j - y_j| : j = 1,2,\cdots\}$

Check D_1: $d(x,x) = \sup\{|x_j - x_j| : j=1,2,\cdots\}$

$= \sup\{0 : j=1,2,\cdots\} = 0$

$d(x,y) = \sup\{|x_j - y_j| : j=1,2,\cdots\}$

Let $x \neq y \in B$, $\exists j$ s.t. $x_j \neq y_j$, thus $|x_j - y_j| > 0$

$d(x,y) > 0$ for distinct $x,y \in B$

Check D_2: $d(x,y) = \sup\{|x_j - y_j| : j=1,2,\cdots\}$

$d(y,x) = \sup\{|y_j - x_j| : j=1,2,\cdots\}$

since $|x_j - y_j| = |y_j - x_j|$, then $\sup\{|x_j - y_j|\} = \sup\{|y_j - x_j|\}$,

thus $d(x,y) = d(y,x)$

Check D_3: $d(x,z) = \sup\{|x_j - z_j| : j=1,2,\cdots\}$

$d(x,y) + d(y,z) = \sup\{|x_j - y_j| : j=1,2,\cdots\} + \sup\{|y_j - z_j| : j=1,2,\cdots\}$

By triangle inequality, $\forall j$ $|x_j - z_j| \leq |x_j - y_j| + |y_j - z_j|$

$d(x,y) + d(y,z) \geq \sup\{|x_j - y_j| + |y_j - z_j| : j=1,2,\cdots\}$

$\geq \sup\{|x_j - z_j| : j=1,2,\cdots\}$

$= d(x,z)$

Therefore, d is a metric for B.

(b) $d^*(x,y) = \sum_{j=1}^{\infty} |x_j - y_j|$

B = the set of all bounded sequences

$d^*(x,y)$ does not define a metric for B

Counterexample:

$x = (1, 1, \cdots)$ $\quad y = (0, 0, \cdots)$

$d^*(x,y) = \sum_{j=1}^{\infty} |1-0| = \infty \notin \mathbb{R}$

13.5 (a) Verify $\bigcap\{S\setminus U : U \in \mathcal{U}\} = S\setminus \bigcup\{U : U\in\mathcal{U}\}$

$$S\setminus \bigcup\{U : U\in\mathcal{U}\} = S\cap\{\bigcup U : U\in\mathcal{U}\}^c$$
$$= S\cap\{\bigcap U^c : U\in\mathcal{U}\}$$
$$= \bigcap\{S\cap U^c : U\in\mathcal{U}\}$$
$$= \bigcap\{S\setminus U : U\in\mathcal{U}\}$$

(b) Show that the intersection of any collection of closed sets is a closed set.

Let $\mathcal{U}$ be any collection of open sets, S be the universal set

$\{S\setminus U : U\in\mathcal{U}\}$ is closed

$\bigcap\{S\setminus U : U\in\mathcal{U}\} = S\setminus\bigcup\{U : U\in\mathcal{U}\}$ by DeMorgen's law

$\bigcup\{U : U\in\mathcal{U}\}$ is a union of open sets, so it is also open,

then $S\setminus\bigcup\{U : U\in\mathcal{U}\}$ is closed

$\bigcap\{S\setminus U : U\in\mathcal{U}\}$ is an intersection of any collection of closed sets and it is closed.

Therefore, the intersection of any collection of closed sets is a closed set.

13.7 Let $U\subseteq\mathbb{R}$ be an open set.

Let $x\in U$. There exists an open interval $I\subset U$ containing x.

Let I_x be the maximal open interval in U containing x (take it to be the union of all open intervals $I\subset U$ containing x).

Such maximal intervals are equal or disjoint: suppose $I_x\cap I_y\neq\emptyset$ and $I_x\neq I_y$, then $I_x\cup I_y$ is an open interval in U containing x, which contradicts that I_x is the maximal interval.

Each of such maximal open interval contains a rational number. $T=\bigcup\limits_{x\in U\cap\mathbb{Q}} I_x$

T is the disjoint union of a collection of open intervals.

4. Given (X, d) a metrix space, ad S a subset of X, we defined the closure of S $\quad \bar{S} = \{p \in X \mid \text{there is a seq. } (p_n) \text{ in } S \text{ that converge to } p\}$

Prove that taking closure again won't make it any bigger, i.e. if $S_1 = \bar{S}$, ad $S_2 = \bar{S_1}$, then $S_1 = S_2$.

Show $\bar{\bar{S}} = \bar{S}$. Need to show $\bar{S} \subset \bar{\bar{S}}$ ad $\bar{\bar{S}} \subset \bar{S}$

If $p \in \bar{S}$, then $\exists (p_n)$ in S s.t. $(p_n) \to p$. Since $S \subset \bar{S}$, $(p_n) \in S$

then $(p_n) \in \bar{S}$, $p \in \bar{\bar{S}}$, thus $\bar{S} \subset \bar{\bar{S}}$.

If $p \in \bar{\bar{S}}$, then $\exists (p_n)$ in $\bar{S}$ s.t. $(p_n) \to p$. $\forall p_n$, $\exists (s_k)$ in S s.t. $(s_k) \to p_n$

$\forall \varepsilon > 0 \quad \exists N_1$ s.t. $\forall n > N_1 \quad d(p_n, p) < \frac{\varepsilon}{2}$

$\forall \varepsilon > 0 \quad \exists N_2$ s.t. $\forall n > N_2 \quad d(s_k, p_n) < \frac{\varepsilon}{2}$

$N = \max\{N_1, N_2\}$

$\forall n > N \quad d(s_k, p) \leq d(s_k, p_n) + d(p_n, p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Thus, $\exists (s_k)$ in S s.t. $(s_k) \to p$

Hence, $p \in \bar{S}$, thus $\bar{\bar{S}} \subset \bar{S}$.

Therefore, $\bar{S} = \bar{\bar{S}}$.

5. Prove that $\overline{S}$ is the intersection of all closed subsets in X that contain S.

WTS for a metric space (X,d) and $\overline{S} \in X$, $\overline{S} = \bigcap \{F \subset X \mid F \text{ is closed}, F \supset S\}$

If $x \in \overline{S}$, suppose $x \notin \bigcap \{F \subset X \mid F \text{ is closed}, F \supset S\}$

Thus, $\exists$ closed $F \supset S$ s.t. $x \notin F$, thus $x \in X \setminus F$

Hence $\overline{S} \subset X \setminus F \Rightarrow S \subset X \setminus F$

However, since $S \subset F$, then $S = \phi$

By contradiction, $x \in \bigcap \{F \subset X \mid F \text{ is closed}, F \supset S\}$

Therefore, $\overline{S} \subset \bigcap \{F \subset X \mid F \text{ is closed}, F \supset S\}$

If $x \in \bigcap \{F \subset X \mid F \text{ is closed}, F \supset S\}$

Then $x \in F$, $x \in S$. Since $S \subset \overline{S}$, then $x \in \overline{S}$.

Therefore $\bigcap \{F \subset X \mid F \text{ is closed}, F \supset S\} \subset \overline{S}$.

Therefore, $\overline{S} = \bigcap \{F \subset X \mid F \text{ is closed}, F \supset S\}$