1.10 n-th proposition:

P_n : $(2n+1)+(2n+3)+\cdots+(4n-1)=3n^2$ for all positive integers n

P_1 : $2\times1+1=3=3\times1^2$

The basis for induction P_1 is true

Suppose P_n is true.

$(2(n+1)+1)+(2(n+1)+3)+\cdots+(4(n+1)-1)$

$=(2n+3)+(2n+5)+\cdots+(4n+3)$

$=(2n+1)+(2n+3)+\cdots+(4n-1)+(4n+1)+2\times(n+1)$

$=3n^2+6n+3$

$=3(n+1)^2$

Thus P_{n+1} holds if P_n holds.

By the principle of methematical induction, P_n is true for all n.

1.12 (a) $n=1$ $(a+b)^1=a+b$

$\binom{1}{0}a^1+\binom{1}{1}a^0b=a+b$

$n=2$ $(a+b)^2=a^2+2ab+b^2$

$\binom{2}{0}a^2+\binom{2}{1}a^1b+\binom{2}{2}a^0b^2=a^2+2ab+b^2$

$n=3$ $(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$\binom{3}{0}a^3+\binom{3}{1}a^2b+\binom{3}{2}a^1b^2+\binom{3}{3}a^0b^3=a^3+3a^2b+3ab^2+b^3$

(b) $$\binom{n}{k}+\binom{n}{k-1}=\frac{n!}{k!(n-k)!}+\frac{n!}{(k-1)!(n-k+1)!}$$

$$=\frac{n!}{k!(n-k)!}+\frac{n!}{k!(n-k)!}\cdot\frac{k}{n-k+1}$$

$$=\frac{n!}{k!(n-k)!}\cdot\left(1+\frac{k}{n-k+1}\right)$$

$$=\frac{n!}{k!(n-k)!}\cdot\frac{n+1}{n-k+1}=\frac{(n+1)!}{k!(n+1-k)!}=\binom{n+1}{k}$$

(c) n^{th} proposition:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n}b^n$$

The basis for induction P_1 is true.

Suppose P_n is true

$$(a+b)^{n+1} = \left[\binom{n}{0}a^n + \cdots + \binom{n}{n}b^n\right](a+b)$$

$$= \binom{n}{0}a^{n+1} + \binom{n}{1}a^n b + \cdots + \binom{n}{n}ab^n$$

$$+ \binom{n}{0}a^n b + \binom{n}{1}a^{n-1}b^2 + \cdots + \binom{n}{n}b^{n+1}$$

$$= \binom{n}{0}a^{n+1} + \left[\binom{n}{0}+\binom{n}{1}\right]a^n b + \cdots + \left[\binom{n}{n-1}+\binom{n}{n}\right]ab^n + \binom{n}{n}b^{n+1}$$

$$= \binom{n}{0}a^{n+1} + \binom{n+1}{1}a^n b + \cdots + \binom{n+1}{n}ab^n + \binom{n}{n}b^{n+1}$$

$$= \binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^n b + \cdots + \binom{n+1}{n}ab^n + \binom{n+1}{n}b^{n+1}$$

Thus P_{n+1} holds if P_n holds.

By the principle of mathematical induction, P_n is true for all n.

2.1 $\sqrt{3}$: The only possible rational solutions of $x^2-3=0$ are $\pm 1, \pm 3$, and none of these numbers are solutions.

$\sqrt{5}$: The only possible rational solutions of $x^2-5=0$ are $\pm 1, \pm 5$, and none of these numbers are solutions.

$\sqrt{7}$: The only possible rational solutions of $x^2-7=0$ are $\pm 1, \pm 7$, and none of these numbers are solutions.

$\sqrt{24}$: The only possible rational solutions of $x^2-24=0$ are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$, and none of these numbers are solutions.

$\sqrt{31}$: The only possible rational solutions of $x^2-31=0$ are $\pm 1, \pm 31$, and none of these numbers are solutions.

2.2 $\sqrt[3]{2}$: The only possible rational solutions of $x^3-2=0$ are $\pm 1, \pm 2$, and none of these numbers are solutions.

$\sqrt[7]{5}$: The only possible rational solutions of $x^7-5=0$ are $\pm1, \pm5$, and none of these numbers are solutions.

$\sqrt[4]{13}$: The only possible rational solutions of $x^4-13=0$ are $\pm1, \pm13$, and none of these numbers are solutions.

2.7 (a) $\sqrt{4+2\sqrt{3}}-\sqrt{3}$
$= \sqrt{(1+\sqrt{3})^2}-\sqrt{3}$
$= 1$

(b) $\sqrt{6+4\sqrt{2}}-\sqrt{2}$
$= \sqrt{(2+\sqrt{2})^2}-\sqrt{2}$
$= 2$

3.1 Theorem:

(i) $a+c=b+c \Rightarrow (a+c)+(-c)=(b+c)+(-c)$
$\Rightarrow a+(c+(-c))=b+(c+(-c))$
$\Rightarrow a+0=b+0$
$\Rightarrow a=b$

(ii) $a\cdot 0 \underset{A3}{=} a\cdot(0+0) \underset{DL}{=} a\cdot 0+a\cdot 0$
$\Rightarrow a\cdot 0+(-a\cdot 0) = a\cdot 0+a\cdot 0+(-a\cdot 0)$
$\Rightarrow 0 = a\cdot 0+(a\cdot 0+(-a\cdot 0))$
$\Rightarrow 0 = a\cdot 0$

(iii) $a+(-a)=0 \Rightarrow ab+(-a)b=(a+(-a))b = 0\cdot b=0$
$\Rightarrow ab+(-ab)=0=ab+(-a)b$
$\Rightarrow (-a)b=-ab$

(iv) $(-a)(-b)+(-a)b = (-a)((-b)+b) = (-a)\cdot 0 = 0$
$\Rightarrow (-a)(-b) = -(-a)b = -(-ab)$
$ab+(-ab)=0 \Rightarrow -(-ab)=ab$

$\Rightarrow (-a)(-b) = ab$

(v) $ac = bc \Rightarrow ac - bc = bc - bc = 0$

$\Rightarrow (a + (-b))c = 0$

$\Rightarrow (a-b)c = 0$

$c \neq 0 \Rightarrow a - b = 0$

$\Rightarrow a = b$

(vi) $ab = 0$ if $b \neq 0$, then $b \cdot b^{-1} = 1$

LHS $= ab \cdot b^{-1} = a(bb^{-1}) = a \cdot 1 = a$

RHS $= 0 \cdot b^{-1} = 0$

$\Rightarrow a = 0$

if $a \neq 0$, then $a \cdot a^{-1} = 1$

LHS $= ab = ba = ba \cdot a^{-1} = b(a \cdot a^{-1}) = b \cdot 1 = b$

RHS $= 0 \cdot a^{-1} = 0$

$\Rightarrow b = 0$

3.2 Theorem:

(i) If $a \leq b$

$a + ((-a)+(-b)) \leq b + ((-a)+(-b))$

$(a+(-a)) + (-b) \leq b + ((-b)+(-a)) = (b+(-b)) + (-a)$

$0 + (-b) \leq 0 + (-a)$

$-b \leq -a$

(ii) If $a \leq b$, $c \leq 0$, then $0 \leq -c$

$\Rightarrow a(-c) \leq b(-c) \Rightarrow -ac \leq -bc$

$\Rightarrow bc \leq ac$

(iii) If $0 \leq a$, $0 \leq b$, then $0 \cdot b \leq a \cdot b \Rightarrow 0 \leq ab$

(iv) If $a \geq 0$, then $a \cdot a \geq 0 \cdot a \Rightarrow a^2 \geq 0$

If $a \leq 0$, then $-a \geq 0 \Rightarrow (-a)\cdot(-a) \geq 0 \cdot (-a) \Rightarrow a^2 \geq 0$

(v) Since $0 \leq a^2$ for all a

Let $a=1$ then $a^2 = 1 \Rightarrow 0 \leq 1$

Remain to show $0 \neq 1$

Suppose for a contradiction that $0 = 1$

Choose $c \in \mathbb{R}$ such that $c \neq 0$

$\Rightarrow 0 \cdot c = 1 \cdot c$ $\quad 0 \cdot c = 0 \quad 1 \cdot c = c \Rightarrow c = 0$ contradiction

Thus, $0 < 1$

(vi) If $0 < a$, suppose for a contradiction that $a^{-1} \leq 0$, then $0 \leq -a^{-1}$

$\Rightarrow 0 \leq a \cdot (-a^{-1}) = -1 \Rightarrow 1 \leq 0$ contradiction

$\Rightarrow 0 < a^{-1}$

(vii) If $0 < a < b$, then $0 < a^{-1}$ $\quad 0 < b^{-1}$

$0 \cdot a^{-1} < a \cdot a^{-1} < b \cdot a^{-1} \Rightarrow 0 < 1 < b \cdot a^{-1}$

$0 \cdot b^{-1} < 1 \cdot b^{-1} < b \cdot a^{-1} b^{-1} = b \cdot b^{-1} a^{-1} \Rightarrow 0 < b^{-1} < a^{-1}$

3.6 (a) $|a+b+c| \leq |a+b| + |c| \leq |a| + |b| + |c|$

(b) n^{th} proposition:

$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$ for n numbers $a_1, \cdots, a_n$

P_1: $|a_1| \leq |a_1|$

The basis for induction P_1 is true.

Suppose P_n is true.

$|a_1 + a_2 + \cdots + a_n + a_{n+1}| \leq |a_1 + \cdots + a_n| + |a_{n+1}|$ by Triangle Inequality

$\leq |a_1| + \cdots + |a_n| + |a_{n+1}|$

Thus P_{n+1} holds if P_n holds.

By the principle of mathematical induction, P_n is true for $a_1, \cdots, a_n$.

4.11 By the Denseness of $\mathbb{Q}$, there is a rational $r_1 \in \mathbb{Q}$ such that $a < r_1 < b$.
Then use the Denseness of $\mathbb{Q}$ again, there is a rational $r_2 \in \mathbb{Q}$ such that $r_1 < r_2 < b$.
Suppose picking rationals as above, there are rationals $r_1, \cdots, r_n \in \mathbb{Q}$ such that $a < r_1 < r_2 < \cdots < r_n < b$.
By Denseness of $\mathbb{Q}$, there is a rational $r_{n+1} \in \mathbb{Q}$ such that $r_n < r_{n+1} < b$.
Thus P_{n+1} holds if P_n holds.
Therefore, there are infinitely many rationals between a and b.

4.14 (a) For all $a \in A$, $a \leq \sup A$
For all $b \in B$, $b \leq \sup B$
$\Rightarrow$ For all $a+b \in A+B$, $a+b \leq \sup A + \sup B$
i.e. $\sup(A+B) \leq \sup A + \sup B$
For any $\varepsilon > 0$, there exists an $a \in A$ and a $b \in B$ such that
$a \geqslant \sup A - \frac{\varepsilon}{2}$ $\quad b \geqslant \sup B - \frac{\varepsilon}{2}$
Thus, $\exists$ $a+b \in A+B$ such that $a+b \geqslant \sup A + \sup B - \varepsilon$
$\Rightarrow \sup(A+B) \geqslant \sup A + \sup B - \varepsilon$
Therefore $\sup(A+B) = \sup A + \sup B$

(b) It is proven in the textbook that $\inf S = -\sup(-S)$
$\inf(A+B) = -\sup(-(A+B)) = -\sup((-A)+(-B)) = (-\sup(-A)) + (-\sup(-B))$
$= \inf A + \inf B$

7.1 (a) $\frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13}, \frac{1}{16}$ $\quad$ (b) $\frac{4}{3}, 1, \frac{10}{11}, \frac{13}{15}, \frac{16}{19}$
(c) $\frac{1}{3}, \frac{2}{9}, \frac{1}{9}, \frac{4}{81}, \frac{5}{243}$ $\quad$ (d) $\frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}$

7.2 (a) converge limit: 0 (b) converge limit: $\frac{3}{4}$

(c) converge limit: 0 (d) not converge

7.3. (a) converge limit: 1 (b) converge limit: 1

(c) converge limit: 0 (d) converge limit: 1

(e) not converge (f) converge limit: 1

(g) not converge (h) not converge

(i) converge limit: 0 (j) converge limit: $\frac{7}{2}$

(k) not converge (l) not converge

(m) converge limit: 0 (n) not converge

(o) converge limit: 0 (p) converge limit: 2

(q) converge limit: 0 (r) converge limit: 1

(s) converge limit: $\frac{4}{3}$ (t) converge limit: 0

7.5. (a) $(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n) = n^2+1-n^2 = 1 \Rightarrow S_n = \frac{1}{\sqrt{n^2+1}+n}$

$$\lim S_n = \lim \sqrt{n^2+1} - n$$

$$= \lim \frac{1}{\sqrt{n^2+1}+n} = 0$$

(b) $(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n) = n^2+n-n^2 = n$

$$\lim(\sqrt{n^2+n}-n) = \lim \frac{n}{\sqrt{n^2+n}+n}$$

$$= \lim \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

(c) $(\sqrt{4n^2+n}-2n)(\sqrt{4n^2+n}+2n) = 4n^2+n-4n^2 = n$

$$\lim(\sqrt{4n^2+n}-2n) = \lim \frac{n}{\sqrt{4n^2+n}+2n}$$

$$= \lim \frac{1}{\sqrt{4+\frac{1}{n}}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$