Why taking this course : — Student Area

critical thinking

coach & tour guide

read sections beforehand

① sequence and limit

Zeno's paradox the sum of infinite segments involved is finite

② topology

abstract concept

open set , closed set

compact $\mathbb{R}$ (closed & bounded)

③ integration & differentiation

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

- Natural number (semi-group)

  $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$

  - successor construction : 2 is the successor of 1

    3 is the successor of 2

    so, starting from 0, one can reach all the natural numbers

  - Peano Axioms for natural number (read : Tao - I)

    mathematical induction property (Axiom 5) :

    Let n be a natural number, $P(n)$ be a statement depending on n, if the following 2 conditions hold

    (a) $P(0)$ is true

    (b) If $P(k)$ is true, then $P(k+1)$ is true. (i.e. $P(k) \Rightarrow P(k+1)$)

then, $P(n)$ is true for all $n \in \mathbb{N}$

- "operations allowed for $\mathbb{N}$": $+$, $\times$ ← "$\cdot$" also use dot for product
  - if $n, m \in \mathbb{N}$, then $n+m \in \mathbb{N}$, $n \times m \in \mathbb{N}$
  - $-$, $/$ are not always defined

- Integers (abelian group)

  $\mathbb{Z} = \{ \cdots, -2, -1, 0, 1, 2, \cdots \}$

  - allowed operation: $+, -, \cdot$ (no division here)

  (formally, $\mathbb{Z}$ is a "ring") (read about "ring" in Harrison Chen's note)

- Rational numbers (field)

  $\mathbb{Q} = \{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \}$

  - we have all 4 operations $+, -, \cdot, /$

  $\mathbb{Q}$ is now a "field".

$\mathbb{Q}$ is an ordered field, there is a "relation" $\leq$.

⌈ A relation S is a subset of $\mathbb{Q} \times \mathbb{Q}$. If $(a,b) \in S$, we say "a and b has relation S", or "$a \, S \, b$". ⌋

The "$\leq$" relation has 3 properties:

(1) $a \leq b$ and $b \leq a$, then $a = b$

(2) if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity)

(3) for any $a, b \in \mathbb{Q}$, we have at least one of the following: $a \leq b$ or $b \leq a$

- Ordered Field $\mathbb{F}$: means the field structure $(+, -, \cdot, /)$ is compatible with $(\leq)$.

(a) if $a \leq b$, then $a + c \leq b + c$ for all $c \in \mathbb{Q}$

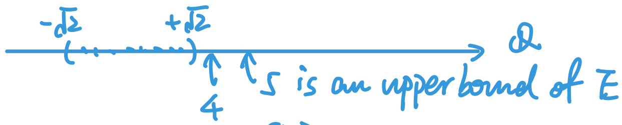
(b) if $a \geq 0$, and $b \geq 0$, then $a \cdot b \geq 0$

What's lacking about $\mathbb{Q}$?

There are certain gaps in $\mathbb{Q}$: for example, the equation $x^2 = 2$ cannot be solved in $\mathbb{Q}$.

For a bounded subset in $\mathbb{Q}$, call it E, it may not have a "most economical" or "sharpest" upperbound in $\mathbb{Q}$.

e.x. $E = \{x \in \mathbb{Q} \mid x^2 < 2\}$



there is no $\overbrace{\text{least upper}}^{\text{sup}}$ bound of E in $\mathbb{Q}$

(we want to say $\sqrt{2}$ as the $\sup(E)$, but $\sqrt{2}$ is not a rational number.)

$\sqrt{2} \neq \frac{m}{n}$

$\mathbb{R}$ is the unique ordered field containing $\mathbb{Q}$ that is complete