Homework:

1. Discord
2. Gradescope
3. StudentArea

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1. Rational Zero Theorem (Ross §2)

Def. An integer coefficient polynomial in x is

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \qquad c_n, \cdots, c_0 \in \mathbb{Z} \quad c_n \neq 0$$

$\mathbb{Z}$-coefficient equation is: $f(x) = 0$

One can ask: when does an $\mathbb{Z}$-coefficient equation has roots in $\mathbb{Q}$

⌈ fact: a degree n polynomial has n roots in $\mathbb{C}$.

i.e. $\exists\, z_1, \cdots, z_n$ in $\mathbb{C}$ such that

$$f(x) = c_n (x - z_1) \cdots (x - z_n)$$ (it is possible that some of the z_i coincide) ⌋

Theorem: If a rational number r satisfies the equation

$$c_n x^n + \cdots + c_1 x^1 + c_0 = 0, \text{ with } c_i \in \mathbb{Z}, c_n \neq 0$$

and $r = \frac{c}{d}$ (where c, d are co-prime integers). Then, c divides c_0, and d divides c_n.

⌈ Ex: (1) $5x + 3 = 0 \qquad r = -\frac{3}{5} \quad c = -3 \quad d = 5$

$c_1 = 5, c_0 = 3$ ⌋

Proof: Plug in $x = \frac{c}{d}$ to equation

$$c_n \left(\frac{c}{d}\right)^n + c_{n-1} \left(\frac{c}{d}\right)^{n-1} + \cdots + c_1 \left(\frac{c}{d}\right) + c_0 = 0$$

Multiply both sides by d^n, we get

$$c_n \cdot c^n + c_{n-1} \cdot c^{n-1} \cdot d + \cdots + c_1 \cdot c \cdot d^{n-1} + c_0 \cdot d^n = 0$$

(1) $\because c_n \cdot c^n = -(c_{n-1} \cdot c^{n-1} \cdot d + \cdots + c_1 \cdot c \cdot d^{n-1} + c_0 \cdot d^n)$

$$= -d(c_{n-1}\cdot c^{n-1} + \cdots + c_0\cdot d^{n-1})$$

$\therefore$ d divides $c_n\cdot c^n$

Since d and c are co-prime, d does not divide c^n

$\therefore$ d has to divide c_n

(2) $c_0\cdot d^n = -(c_n\cdot c^n + c_{n-1}c^{n-1}\cdot d + \cdots + c_1\cdot c\cdot d^{n-1})$

$$= -c(c_n\cdot c^{n-1} + c_{n-1}c^{n-2}\cdot d + \cdots + c_1\cdot d^{n-1})$$

by similar reasoning, $c \mid c_0$. $\square$

Using this rational zero theorem, we can answer questions

claim: (Ex 4)

$\sqrt[3]{6}$ is not rational number $\Leftrightarrow$ $x^3-6=0$ does not have rational roots.

Pf: The only possible rational solution $r=\frac{c}{d}$ needs

$c \mid 6$, $d \mid 1$ $\quad\therefore$ take $d=1$, $c=\pm1, \pm2, \pm3, \pm6$.

One can test all of them, they don't solve the equation

$\therefore$ There is no solution in $\mathbb{Q}$. $\square$

- Real numbers

Historical construction of $\mathbb{R}$ from $\mathbb{Q}$:

(1) Dedekind cut: ($\mathbb{Q}$: if $\sqrt{2} \notin \mathbb{Q}$, how to "save the info" of $\sqrt{2}$?)

$C_{\sqrt{2}} = \{r\in\mathbb{Q} \mid r<\sqrt{2}\}$ a subset

moral: for each $\alpha\in\mathbb{R}$, consider $C_\alpha = \{r\in\mathbb{Q} \mid r<\alpha\}$

one can define addition, multiplication on these subsets C_α.

(2) Sequence in $\mathbb{Q}$

i.e. to use a sequence of rational numbers to "approximate" a real number.

e.g. $\sqrt{2}$ can be approximated by

$1, 1.4, 1.41, 1.414, \cdots\cdots$

problem here: ① given any real numbers, how do you get such a sequence?

② how to tell if 2 different sequences approximate the same real number.

(e.g. $1 \leftarrow 1.1, 1.01, 1.001, \cdots\cdots$

$1 \leftarrow 0.9, 0.99, 0.999, \cdots$

or $1 \leftarrow 1, 1, 1, \cdots$)

The 3 sequences all have the same limit (What is a limit?)

- Given the existence of $\mathbb{R}$, we have properties (axioms) of $\mathbb{R}$
  defining properties
  - completeness of $\mathbb{R}$:

    Given any subset $E \subset \mathbb{R}$, bounded above,

    > bounded above:
    > $\exists\, a \in \mathbb{R}$, such that for any $x \in E$,
    > we have $x \leq a$

    there exist a unique r, $r \in \mathbb{R}$

    ① r is an upper bound of E

    ② for any other upper bound α, we have $r \leq \alpha$.

    r is called the least upper bound of E, $r = \sup E$.

    (i.e. $\sup(E)$ is well-defined for subset E that's bounded above)

    Ex. $\sup([0,1]) = 1 \qquad \sup((0,1)) = 1$

    $\sup(\{r \in \mathbb{Q} \mid r^2 < 2\}) = \sqrt{2}$

  - Corollary: (Archaemedian property): For any $r \in \mathbb{R}$, $r > 0$,

    $\exists\, n \in \mathbb{N}$, such that $n \cdot r > 1 \iff r > \frac{1}{n}$

$+\infty, -\infty$

- With these symbols introduced, we can say

  $\sup(\mathbb{N}) = +\infty \iff \mathbb{N}$ is not bounded above

- $+\infty, -\infty$ are not real numbers. They have partly the operations that $\mathbb{R}$ has i.e. $3 \cdot (+\infty) = +\infty$, $(-3)\cdot(+\infty) = -\infty$.

but $(-\infty)+(-\infty) \neq$ NAN $\quad (0)\cdot(+\infty) =$ not defined

- Sequences and Limits:
  - a sequence of real number, $a_0, a_1, a_2, \cdots$ denoted as $(a_n)_{n=0}^{\infty}$ or shortened (a_n)

    note, use $(\cdots)$, not $\{\cdots\}$

  - We only care about the "eventual behavior" of a sequence
  - Def: A sequence (a_n) converge to $a \in \mathbb{R}$, if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that, $\forall n > N$, $|a_n - a| < \varepsilon$.