Sequence and Limits

Ross §9

Recall last time:

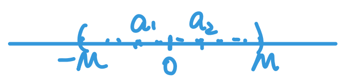
Def: A sequence of real numbers (a_n) converges to $a \in \mathbb{R}$, if for any $\varepsilon > 0$, there exists N (positive integer), such that for all $n > N$, we have $|a_n - a| < \varepsilon$

Today: properties for convergent sequence

Theorem 9.1 Convergent sequences are bounded

recall: a sequence (a_n) is bounded, if $\exists M > 0$,

$|a_n| \le M$ for all n



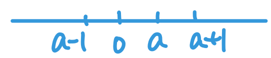
Proof: Let (a_n) be a convergent sequence, that converge to $a \in \mathbb{R}$.

Let $\varepsilon = 1$, then by definition of convergence, there exists $N > 0$, s.t.

$\forall n > N$, $|a_n - a| < 1 \Leftrightarrow a - 1 < a_n < a + 1 \quad \forall n > N$

Let $M_1 = \max\{a_1, a_2, \cdots, a_N\}$

$M_2 = \max\{|a-1|, |a+1|\}$



$M = \max\{M_1, M_2\}$

Thus, if $n \le N$, we have $|a_n| \le M_1$

if $n > N$, we have $|a_n| \le M_2$

Thus, $\forall n$, $|a_n| \le \max\{M_1, M_2\} = M$

Moral: one can deal with the first few terms of a sequence easily, & it is the "tail of the sequence" that matters.

Operations on convergent sequence:

(1) • $\forall c \in \mathbb{R}$, $\forall$ convergent seq. $a_n \to a$, we have

$c \cdot a_n \to c \cdot a$

Pf: If $c=0$, the result is obvious.

If $c\neq 0$, we need to show for any $\varepsilon>0$, $\exists N>0$.

s.t. $\forall n>N$ $\quad |c\cdot a_n - ca|<\varepsilon$. But, this is equivalent to

$|c|\cdot|a_n-a|<\varepsilon$

$\Leftrightarrow \quad |a_n-a|<\frac{\varepsilon}{|c|}$

Now, let $\varepsilon'=\frac{\varepsilon}{|c|}$ and using convergence of $a_n\to a$, we have

$N>0$ such that $\forall n>N$, $|a_n-a|<\varepsilon'=\frac{\varepsilon}{|c|}$.

This gives us the desired N. $\square$

(2) • If $a_n\to a$, $b_n\to b$, then $a_n+b_n\to a+b$.

Pf: We want to show. $\forall\varepsilon>0$, $\exists N$ such that

$\forall n>N \quad |a_n+b_n-(a+b)|<\varepsilon$

The requirement $\Leftrightarrow$ $|(a_n-a)+(b_n-b)|<\varepsilon \qquad (*)$

$\because$ $|(a_n-a)+(b_n-b)|\leq|a_n-a|+|b_n-b|$ by triangle inequality

$\therefore$ $(*) \Leftarrow |a_n-a|+|b_n-b|<\varepsilon \qquad (**)$

$$\Leftarrow \begin{cases}|a_n-a|<\frac{\varepsilon}{2}\\ |b_n-b|<\frac{\varepsilon}{2}\end{cases} \qquad (***)$$

By convergence of a_n and b_n, $\exists N_1, N_2$, such that

$\forall n>N_1 \quad |a_n-a|<\frac{\varepsilon}{2}$; $\forall n>N_2$, $|b_n-b|<\frac{\varepsilon}{2}$

Take $N=\max\{N_1,N_2\}$, then $\forall n>N$. $(***)$ is satisfied.

Hence, $(*)$ is satisfied.

• Cor: If $a_n\to a$, $b_n\to b$, then $a_n-b_n\to a-b$

Pf: Let $c_n=(-1)\cdot b_n$, then $c_n\to -b$.

then $a_n-b_n=a_n+c_n\to a-b$.

(3) • If $a_n\to a$, $b_n\to b$, then $a_n\cdot b_n\to a\cdot b$.

Pf: WTS: $\forall \varepsilon > 0, \exists N.$ such that $\forall n > N$

$$|a_n b_n - ab| < \varepsilon \qquad (*)$$

$$|a_n b_n - ab| = |a_n(b_n - b) + a_n \cdot b - ab|$$
$$= |a_n(b_n-b) + (a_n - a)b|$$
$$\le |a_n(b_n-b)| + |(a_n-a)b|$$
$$\le |a_n||b_n-b| + |a_n-a||b|$$

($\because a_n \to a$ $\therefore a_n$ is bounded $\therefore \exists M_1 > 0$ s.t. $|a_n| \le M_1$ $\forall n$)

$$\le M_1|b_n-b| + |b|\cdot|a_n-a|$$

$$(*) \Leftarrow \begin{cases} M_1 \cdot |b_n - b| < \frac{\varepsilon}{2} \\ |b|\cdot|a_n-a| < \frac{\varepsilon}{2} \end{cases} \qquad (**)$$

Since $a_n \to a$, let $\varepsilon_1 = \frac{\varepsilon}{2}/|b|$, then $\exists N_1$, s.t. $\forall n > N_1$,

$|a_n - a| < \varepsilon_1 \Leftrightarrow |b|\cdot|a_n-a| < \frac{\varepsilon}{2}$.

Since $b_n \to b$, let $\varepsilon_2 = \frac{\varepsilon}{2}/M_1$, then $\exists N_2$, s.t. $\forall n > N_2$,

$|b_n - b| < \varepsilon_2 \Leftrightarrow M_1 \cdot |b_n - b| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$, thus for all $n > N$,

we have $(**)$ holds, hence $(*)$ holds. $\square$

(4) • If $a_n \to a$, and $a_n \ne 0$ $\forall n$, and $a \ne 0$, then $\frac{1}{a_n} \to \frac{1}{a}$.

(note, $a_n \ne 0$, does not imply $a \ne 0$, ex. $a_n = \frac{1}{n}, a = 0$)

Pf: WTS: $\forall \varepsilon > 0, \exists N$, s.t. $\forall n > N$

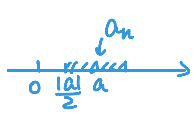
$$\left|\frac{1}{a_n} - \frac{1}{a}\right| < \varepsilon \qquad (*)$$

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a - a_n}{a \cdot a_n}\right| = \frac{|a - a_n|}{|a||a_n|}$$

Claim: $\exists c > 0$, such that $|a_n| > c$ $\forall n$.

Let $\varepsilon' = \frac{|a|}{2}$, then $\exists N'$, s.t. $\forall n > N'$ $|a_n - a| < \varepsilon' = \frac{|a|}{2}$

$\Leftrightarrow \quad -\frac{|a|}{2} < a_n - a < \frac{|a|}{2}$



$\Leftrightarrow \quad a - \frac{|a|}{2} < a_n < a + \frac{|a|}{2}$

If $a>0$, then $|a| = a$. $\frac{1}{2}|a| < a_n < \frac{3}{2}|a|$

If $a<0$, then $a = -|a|$ $\quad -\frac{3}{2}|a| < a_n < -|a| + \frac{|a|}{2} = -\frac{1}{2}|a|$

$\Rightarrow |a_n| > \frac{1}{2}|a|$

$\Rightarrow \quad |a_n| > \frac{|a|}{2}$

let $c_1 = \min\{|a_1|, |a_2|, \cdots, |a_{N'}|\} > 0$

let $c = \min\{c_1, \frac{|a|}{2}\}$ finish proof of claim

Thus, $\frac{|a_n - a|}{|a| \cdot |a_n|} < \frac{|a_n - a|}{|a| \cdot c}$

Hence, $(*) \Leftarrow \frac{|a_n - a|}{|a| \cdot c} < \varepsilon \quad (**)$

and $(**)$ can be achieved using $a_n \to a$ $\square$

- Cor: If $a_n \to a$, $b_n \to b$, and $b_n \neq 0$, $b \neq 0$,

  then $\frac{a_n}{b_n} \to \frac{a}{b}$

Pf: $\frac{a_n}{b_n} = a_n \cdot (\frac{1}{b_n})$ $\quad \because \frac{1}{b_n} \to \frac{1}{b}$ by Lemma.

$\therefore a_n \cdot (\frac{1}{b_n}) \to a \cdot \frac{1}{b}$ ($\because$ product of convergent sequence still converge) $\square$

Theorem 9.7: (Useful Results)

(1) $\lim_{n\to\infty} \frac{1}{n^p} = 0 \quad \forall p > 0$

(2) $\lim_{n\to\infty} a^n = 0 \quad \forall |a| < 1$

(3) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

sketch: let $s_n = n^{\frac{1}{n}} - 1$, thus $s_n \geq 0$ $\forall n$ positive integer.

$1 + s_n = n^{\frac{1}{n}} \Leftrightarrow (1+s_n)^n = n$

using binomial expansion

$1 + n \cdot s_n + \frac{n(n-1)}{2} s_n^2 + \cdots = n$

$\Rightarrow \frac{n(n-1)}{2} \cdot s_n^2 \leq n$

$\Rightarrow s_n^2 \leq \frac{2}{n-1}$

Thus $s_n \to 0$ as $n \to \infty$

(4) $\lim a^{\frac{1}{n}} = 1 \quad \forall a > 0$

$$\lim a^{\frac{1}{n}} = a^{\lim \frac{1}{n}} = a^0 = 1$$

Discussion. Ross Exercise 9.2, 9.9(c), 9.15

9.2 (a) $\lim(x_n + y_n) = \lim x_n + \lim y_n = 3 + 7 = 10$

(b) $\lim \frac{3y_n - x_n}{y_n^2} = \lim(3y_n - x_n) \cdot \lim \frac{1}{y_n^2} = (3\lim y_n - \lim x_n) \frac{1}{\lim y_n \lim y_n} = \frac{3\times 7 - 3}{7\times 7} = \frac{18}{49}$

9.9 (c) If limits exist $\forall\, \varepsilon_1, \varepsilon_2$, $\exists\, N_1, N_2$ s.t.

$|s_n - s| < \varepsilon_1 \quad \forall n > N_1$

WTS: $s \leq t$

$|t_n - t| < \varepsilon_2 \quad \forall n > N_2$

$s_n \leq t_n \quad \forall n > N_0 \quad \Rightarrow \quad t_n - s_n \geq 0 \quad \forall n > N_0$

$\Rightarrow \lim(t_n - s_n) \geq 0$

$\Rightarrow \lim t_n - \lim s_n \geq 0 \quad \Rightarrow \quad \lim s_n \leq \lim t_n$

?

9.15 $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$

Let $s_n = \frac{a^n}{n!}$

$\left|\frac{s_{n+1}}{s_n}\right| = \frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n} = \frac{a}{n+1}$

$\lim \left|\frac{s_{n+1}}{s_n}\right| = \lim \frac{a}{n+1} = a \lim \frac{1}{n+1} = 0 < 1$

Since $\lim \left|\frac{s_{n+1}}{s_n}\right| = L$ exists and $L < 1$, then $\lim s_n = 0$.

9.9. (c) Let $s_n \to s$, $t_n \to t$

Suppose $s > t$ $\forall n > N_0$. Let $\varepsilon = \frac{s-t}{2}$

Since limits exist $\forall\, \varepsilon_1, \varepsilon_2$, $\exists\, N_1, N_2$ s.t.

$|s_n - s| < \varepsilon \quad \forall n > N_1 \quad \Rightarrow \quad s - \varepsilon < s_n < s + \varepsilon \quad \forall n > N_1$

$|t_n - t| < \varepsilon \quad \forall n > N_2 \quad \Rightarrow \quad t - \varepsilon < t_n < t + \varepsilon \quad \forall n > N_2$

Let $N = \max\{N_0, N_1, N_2\}$

Then, $\forall n > N$, we have $t_n < t + \varepsilon = t + \frac{s-t}{2} = \frac{s+t}{2} = s - \varepsilon < s_n$

which contradicts the assumption $s_n \leq t_n$ $\forall n > N_0$