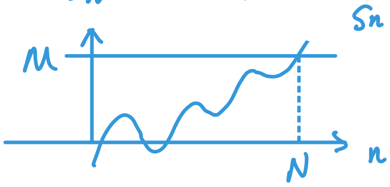
Ross §10 : monotone sequence & lim sup, lim inf

- Def ($\lim s_n = +\infty$): A sequence (s_n) is said to "diverge to $+\infty$", if for any $M \in \mathbb{R}$, there is an N s.t.

$$s_n > M \quad \forall n > N$$



- Recall:
- Def (sup of a set): Given a subset $S \subset \mathbb{R}$. If S is not bounded above, then $\sup S = +\infty$. If S is bounded above, then $\sup S$ is the number r, that is an upper bound, and for any $\varepsilon > 0$, there is some $s \in S$, that $s > r - \varepsilon$ ($r - \varepsilon$ is not an upper bound of S)
- Def (value set of a sequence): If $(s_n)_{n=1}^{\infty}$ is a sequence, then $\{s_n\}_{n=1}^{\infty}$, the subset of $\mathbb{R}$ that (s_n) values in, is called the value set.

Ex: • $(s_n) = 1, 2, 1, 2, 1, 2, \cdots$ "journey"

$\{s_n\}_{n=1}^{\infty} = \{1, 2\}$ "foot print"

• $(s_n) = 1, 1, 2, 2, 1, 1, 2, 2, \cdots$

$\{s_n\}_{n=1}^{\infty} = \{1, 2\}$

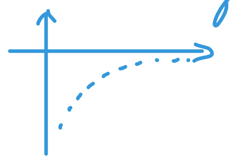
• $(s_n) = 1, 2, 3, 4, \cdots$

$\{s_n\}_{n=1}^{\infty} = \{1, 2, 3, 4, \cdots\}$

- Def (monotone sequence)
  - A seq$_n$ $(a_n)_{n=1}^{\infty}$ is (weakly) monotonically increasing, if $a_{n+1} \geq a_n \quad \forall n = 1, 2, \cdots$
  - A seq$_n$ $(a_n)_{n=1}^{\infty}$ is monotonically decreasing, if $a_{n+1} \leq a_n \quad \forall n = 1, 2, \cdots$

Ex: • $(a_n) = a$, constant sequence is monotone increasing & decreasing

- $(a_n) = 1, 2, 3, \cdots$ is increasing
- $(a_n) = -\frac{1}{n}$, $n \geq 1$, $\lim a_n = 0$



increasing, and bounded above (hence, bounded below)

<u>Theorem</u>: A bounded monotone sequence is convergent.

<u>Pf</u>: (only prove for increasing sequence).

Let a_n be a bounded monotone sequence. Let $\gamma = \overbrace{\sup\{a_n\}_{n=1}^{\infty}}^{\text{a set}}$ $(= \sup_n a_n)$.

Then, • $a_n \leq \gamma \quad \forall n$

• for any $\varepsilon > 0$, $\exists\, a_{n_0}$, s.t. $a_{n_0} > \gamma - \varepsilon$

Thus, for any $\varepsilon > 0$, let $N = n_0$ (n_0 defined above), then for any $n > N$, we have

$$\gamma - \varepsilon < a_{n_0} \leq a_n \leq \gamma$$

thus $$|a_n - \gamma| < \varepsilon$$

Hence, $\lim a_n = \gamma$. $\square$

<u>Ex</u>: (Recursively defined sequence):

Let S_1 be any positive number. Let

$$(*) \qquad S_{n+1} = \frac{S_n^2 + 5}{2S_n} \qquad \forall n \geq 1$$

We want to show $\lim S_n$ exists and find it.

<u>Remark</u>: (1) if we assume $\lim S_n$ exists, call it S, then S satisfies

$$(**) \qquad S = \frac{S^2+5}{2S}$$

$\because$ We can apply the operation $\lim\limits_{n\to\infty}(\cdots)$ to both sides of $(*)$

$(**) \Rightarrow 2S^2 = S^2 + 5 \Rightarrow S^2 = 5 \Rightarrow S$ can be $\pm\sqrt{5}$

Since S_n is a positive sequence, $\lim S_n$ can only be ≥ 0,

thus S can only be $\sqrt{5}$.

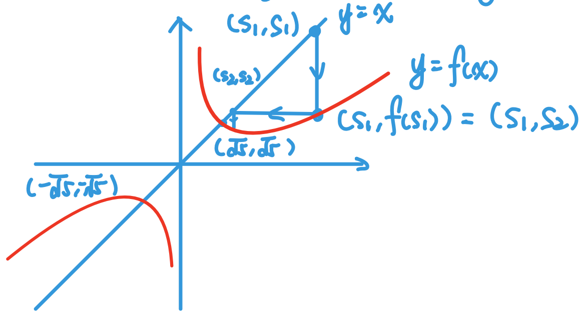
(2) To show $\lim S_n$ exists, we only need to show S_n is bounded

and monotone.

Here is a trick: • let $f(x) = \frac{x^2+5}{2x}$, then $s_{n+1} = f(s_n)$

• Consider the graph of f, i.e. $y = f(x)$

• Consider the diagonal, i.e. $y = x$

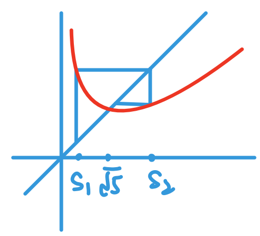


(1) • if $s_1 > \sqrt{5}$, we should try to prove

$\sqrt{5} < \cdots s_3 < s_2 < s_1$

(2) • if $0 < s_1 < \sqrt{5}$, then we have $s_2 > \sqrt{5}$,

we can consider $(s_n)_{n=2}^{\infty}$, that reduces to case (1).



• If (s_n) is unbounded an increasing, then $\lim s_n = +\infty$.

If (s_n) is unbounded an decreasing, then $\lim s_n = -\infty$.

§ Lim inf and lim sup of a sequence

• Def (lim sup): Let $(s_n)_{n=1}^{\infty}$ be a sequence,

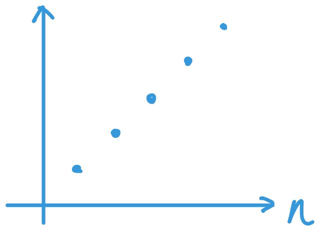
$$\varlimsup_{n\to\infty} s_n = \lim_{n\to\infty} \left(\sup \{s_m\}_{m=n}^{\infty}\right)$$

Notation: • $(s_n)_{n=N}^{\infty}$ is called a "tail of the sequence (s_n)" starting at N.

• $A_N = \sup \{s_n\}_{n=N}^{\infty} = \sup_{n \geq N} s_n$.

• $\limsup s_n = \lim_{N\to\infty} A_N$

Ex. (1) $(s_n) = 1, 2, 3, 4, 5, \cdots$

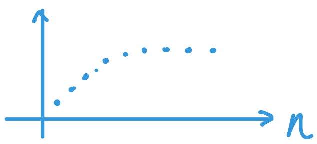


$A_1 = \sup_{n \geq 1} S_n = +\infty$

$A_2 = \sup_{n \geq 2} S_n = +\infty$

$\limsup S_n = \lim A_N = +\infty$

(2) $(S_n) = 1 - \frac{1}{n}$



$A_1 = \sup_{n \geq 1} S_n = 1$

$A_2 = \sup_{n \geq 2} S_n = 1$

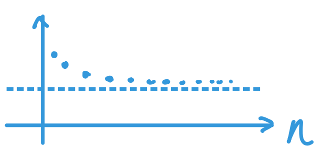
$\vdots$

$A_n = 1$

$\limsup S_n = \lim A_N = 1$

(actually, for any monotone decreasing seq, $\limsup S_n = \sup S_n = A_1$)

(3) $(S_n) = 1 + \frac{1}{n}$



$(S_n) = 2,\ 1+\frac{1}{2},\ 1+\frac{1}{3},\ 1+\frac{1}{4},\ \cdots$

$A_1 = \sup\{2, 1+\frac{1}{2}, 1+\frac{1}{3}, \cdots\} = 2$

$A_2 = \sup\{1+\frac{1}{2}, 1+\frac{1}{3}, \cdots\} = 1+\frac{1}{2}$

$\vdots$

$A_n = S_n = (1+\frac{1}{n})$

$\limsup S_n = \lim(1+\frac{1}{n}) = 1$

<u>Lemma</u>: $A_n = \sup_{m \geq n} S_m$ forms a decreasing sequence.

Pf: Since $\{S_m\}_{m=n}^{\infty} \supset \{S_m\}_{m=n+1}^{\infty}$, thus

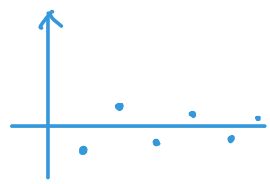
$$\sup\{S_m\}_{m=n}^{\infty} \geq \sup\{S_m\}_{m=n+1}^{\infty},$$

i.e. $A_n \geq A_{n+1}$

Corollary: $\lim\limits_{n\to\infty} A_n = \inf\{A_n\}_{n=1}^{\infty}$ $(= \inf_n A_n)$

(4) $S_n = (-1)^n \cdot \frac{1}{n}$

$S_1 = -1$, $S_2 = \frac{1}{2}$, $S_3 = -\frac{1}{3}$, $\cdots$



$A_1 = \sup_{n\geq 1}(S_n) = S_2 = \frac{1}{2}$

$A_2 = \frac{1}{2}$

$A_3 = S_4 = \frac{1}{4}$

$(A_n) = \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \cdots$

A_n is like the "upper envelope".

$\limsup S_n = \lim A_n = 0$

Ex.

10.1

	increasing	decreasing	bounded
(a) $\frac{1}{n}$		✓	✓
(b) $\frac{(-1)^n}{n^2}$			✓
(c) n^5	✓		
(d) $\sin(\frac{n\pi}{7})$			✓
(e) $(-2)^n$			
(f) $\frac{n}{3^n}$		✓	✓