Homework 1

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Problem 1 (Ross 1.10)**.** *For any $n \in \mathbb{N}$,*

$$\sum_{k=1}^{n}(2n+2k-1) = 3n^2.$$

Proof. Let

$$A = \{n \in \mathbb{N} \mid \sum_{k=1}^{n}(2n+2k-1) = 3n^2\}.$$

We will show that $0 \in A$ and if $n \in A$, then $n+1 \in A$. First, by convention,

$$\sum_{k=1}^{0}(2n-1+2k) = 0$$

and observe that $3(0)^2 = 0$. Thus, $0 \in A$.

Now, for any $n \in \mathbb{N}$, observe that

$$\begin{aligned} 3n^2 &= \sum_{k=1}^{n}(2n+2k-1) \\ &= \sum_{k=0}^{n-1}(2n+2(k+1)-1) \\ &= \sum_{k=0}^{n-1}(2n+2k+1) \\ &= (2n+1) + \sum_{k=1}^{n-1}(2n+2k+1) \\ 3n^2 + (4n+1) &= (2n+1) + \sum_{k=1}^{n}(2n+2k+1) \\ 3n^2 + (4n+1) + (4n+3) &= (2n+1) + \sum_{k=1}^{n+1}(2n+2k+1) \\ 3n^2 + (4n+1) + (4n+3) - (2n+1) &= \sum_{k=1}^{n}(2n+2k+1). \end{aligned}$$

Now we have

$$3n^2 + (4n+1) + (4n+3) - (2n+1) = 3n^2 + 6n + 3 = 3(n+1)^2.$$

Thus, $3(n+1)^2 = \sum_{k=1}^{n}(2n+2k+1)$ meaning that $n+1 \in A$. Thus, $A = \mathbb{N}$ meaning this statement is true for all natural numbers n. ☐

Note that if the reader is uncomfortable with the summation convention used, we could also let A be the set of natural numbers n that satisfy the equation or $n = 0$. In this case, proving the inductive step would be done in two cases, first when n is not 0, and second when $n = 0$ (which is equivalent to proving the base case if we were inducting over the set of positive integers).

Problem 2 (Ross 1.12)**.** *For any natural number $n \in \mathbb{N}$, the following is true for all natural numbers n and $a, b \in R$, where R is a commmutative ring:*

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

(Note that $\binom{n}{i}$ represents repeated addition of $\binom{n}{i}$ copies of $a^i b^{n-i}$. It is thus an integer and not necessarily an element of R in which the explicit formula may not hold in the case that R has nonzero characteristic. Alternatively, we can consider $\binom{n}{i}$ here to be the map of $\binom{n}{i} \in \mathbb{Z}$ into the copy of $\mathbb{Z}/k\mathbb{Z} \subseteq R$ given by the canonical projection.)

2.a. *The binomial theorem holds for $n = 0, 1, 2, 3$.*

Proof. We have $(a+b)^0 = 1 = \binom{0}{0} a^0 b^0$ and $(a+b)^1 = \binom{1}{0} a + \binom{1}{1} b$ which are the exansions of the summations for $n = 0$ and $n = 1$. For $n = 2$, we have

$$\begin{aligned}
(a+b)^2 &= (a+b)(a+b) \\
&= a(a+b) + b(a+b) \\
&= a^2 + ab + ba + b^2 \\
&= a^2 + ab + ab + b^2 \\
&= a^2 + 2ab + b^2 \\
&= \binom{2}{0} a^2 + \binom{2}{1} ab + \binom{2}{2} b^2.
\end{aligned}$$

For $n = 3$:

$$\begin{aligned}
(a+b)^3 &= (a+b)^2(a+b) \\
&= (a^2 + 2ab + b^2)(a+b) \\
&= (a^2 + 2ab + b^2)a + (a^2 + 2ab + b^2)b \\
&= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 \\
&= a^3 + 3a^2b + 3ab^2 + b^3.
\end{aligned}$$

□

2.b. *For any integers $0 < k < n$,*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof. Observe

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!} \left(\frac{1}{k} + \frac{1}{n-k} \right) \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!} \left(\frac{n-k}{k(n-k)} + \frac{k}{k(n-k)} \right) \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!} \left(\frac{n}{k(n-k)} \right) \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}.
\end{aligned}$$

□

The other facts we need is that $\binom{n}{0} = \binom{n}{n} = 1$. This follows from the definition:

$$\binom{n}{0} = \frac{n!}{n!0!} = 1$$

and

$$\binom{n}{n} = \frac{n!}{0!n!} = 1.$$

2.c. *For any natural number n and $a, b \in R$,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Proof. Let:

$$A = \{n \in \mathbb{N} \mid (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\}.$$

Observe that $0, 1, 2, 3 \in A$ as we have already shown the binomial theorem holds for them. Now we show that if $n \in A$, then $n+1 \in A$. Let $n \in A$. Thus:

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)^n(a+b) \\
&= \left(\sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}\right)(a+b) \\
&= \left(\sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}\right) a + \left(\sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}\right) b \\
&= \sum_{i=0}^{n} \binom{n}{i} a^{i+1} b^{n-i} + \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i+1} \\
&= \sum_{i=1}^{n+1} \binom{n}{i-1} a^i b^{n-i+1} + \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i+1} \\
&= \binom{n}{n} a^{n+1} + \sum_{i=1}^{n} \binom{n}{i-1} a^i b^{n-i+1} + \sum_{i=1}^{n} \binom{n}{i} a^i b^{n-i+1} + \binom{n}{0} b^{n+1} \\
&= a^{n+1} + \sum_{i=1}^{n} \left(\binom{n}{i-1} + \binom{n}{i}\right) a^i b^{n-i+1} + b^{n+1} \\
&= \binom{n+1}{n+1} a^{n+1} + \sum_{i=1}^{n} \binom{n+1}{i} a^i b^{n-i+1} + \binom{n+1}{0} b^{n+1} \\
&= \sum_{i=0}^{n+1} \binom{n+1}{i} a^i b^{n-i+1}.
\end{aligned}$$

Thus, the theorem holds for $n+1$, meaning that $n+1 \in A$. Thus, $A = \mathbb{N}$ implying that the binomial theorem holds for all natural numbers. □

Problem 3 (Ross 2.1)**.** *The numbers $\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{24}, \sqrt{31} \in \mathbb{R}$ are not rational numbers.*

Proof. Observe that $\sqrt{3}$ is a root of the polynomial $x^2 - 3$ over $\mathbb{R}$. We claim this polynomial has no rational solutions. Observe that as it has integer coefficients, by the rational root theorem, the only rational solutions are $\pm 1, \pm 3$. Notice as $(\pm 1)^2 - 3 = -2 \neq 0$ and $(\pm 3)^2 - 3 = 6 \neq 0$ (as $\mathbb{Q}$ and $\mathbb{R}$ have characteristic 0), $x^2 - 3$ has no rational solutions. So, $\sqrt{3} \notin \mathbb{Q}$.

Likewise, note that $\sqrt{5} \in \mathbb{R}$ satisfies $x^2 - 5 = 0$. By the Rational Root Theorem, the rational roots of $x^2 - 5$ are contained in $\{\pm 1, \pm 5\}$. However, ± 1 is not a solution as $1 - 5 = -4 \neq 0$ and ± 5 is not a solution either as $25 - 5 = 20 \neq 0$. Thus, $\sqrt{5} \notin \mathbb{Q}$.

Similarly $\sqrt{7} \in \mathbb{R}$ is a root of $x^2 - 7$. By the Rational Root Theorem, solutions to this polynomial are contained in $\{\pm 1, \pm 7\}$. However, neither ± 1 nor ± 7 are roots as $1 - 7 = -6 \neq 0$ and $49 - 7 = 42 \neq 0$. As $x^2 - 7$ has no rational solutions, $\sqrt{7}$ cannot be rational.

For $\sqrt{24} \in \mathbb{R}$, observe that $\sqrt{24} = 2\sqrt{6}$ so if $\sqrt{24} \in \mathbb{Q}$, then as $\frac{1}{2} \in \mathbb{Q}$, $\sqrt{6} \in \mathbb{Q}$. However, $\sqrt{6}$ is a root of the polynomial $x^2 - 6$ which, by the Rational Root Theorem, only has solutions contained in $\{\pm 1, \pm 2, \pm 3, \pm 6\}$. However, none of these four solutions are roots, so $\sqrt{6} \notin \mathbb{Q}$ and thus $\sqrt{24} \notin \mathbb{Q}$.

Finally, for $\sqrt{31}$, observe it is a root of the polynomial $x^2 - 31$. This polynomial has solutions contained in $\{\pm 1, \pm 31\}$, neither of which are solutions; thus, $\sqrt{31}$ is irrational as well. $\square$

Observe we can extend this technique to any squarefree number, showing that in general, square roots of non-square integers are irrational.

Problem 4 (Ross 2.2)**.** *The numbers* $\sqrt[3]{2}$, $\sqrt[7]{5}$, $\sqrt[4]{13} \in \mathbb{R} \setminus \mathbb{Q}$.

Proof. We assume these numbers are contained in $\mathbb{R}$ (which can be proven by showing $x \mapsto x^n$ is a continuous function and applying the Intermediate Value Theorem). We now show they are irrational.

Observe $\sqrt[3]{2}$ is a root of $x^3 - 2$. By the Rational Root Theorem, the solutions to this polynomial are contained in $\{\pm 1, \pm 2\}$; since these are not solutions, $x^3 - 2$ has no rational solutions and $\sqrt[3]{2}$ is irrational.

Next, $\sqrt[7]{5}$ is a root of $x^7 - 5$. By the Rational Root Theoreme, the solutions to this polynomial arae contained in $\{\pm 1, \pm 5\}$; however, neither are solutions as $\pm 1 - 5 \neq 0$ and $\pm 5^7 - 5 \neq 0$ as $5^7 > 5$ and $-5^7 < 5$. So, as $x^7 - 5$ has no rational roots, $\sqrt[7]{5}$ is irrational.

Finally, $\sqrt[4]{13}$ is a root of $x^4 - 13$; by the Rational Root Theorem, solutions to this polynomial are contained in $\{\pm 1, \pm 13\}$. Since these are not solutions, as $1 - 13 \neq 0$ and $13^4 > 13$, $\sqrt[4]{13}$ is irrational. $\square$

Problem 5. *Ross 2.7 The real numbers* $\sqrt{4 + 2\sqrt{3}} - \sqrt{3}$ *and* $\sqrt{6 + 4\sqrt{2}} - \sqrt{2}$ *are rational.*

Proof. We have:

$$\begin{aligned} \sqrt{4 + 2\sqrt{3}} - \sqrt{3} &= \sqrt{3 + 1 + 2\sqrt{3}} - \sqrt{3} \\ &= \sqrt{(1 + \sqrt{3})^2} - \sqrt{3} \\ &= |1 + \sqrt{3}| - \sqrt{3} \\ &= 1 + \sqrt{3} - \sqrt{3} \\ &= 1. \end{aligned}$$

Similarly:

$$\begin{aligned} \sqrt{6 + 4\sqrt{2}} - \sqrt{2} &= \sqrt{4 + 2 + 4\sqrt{2}} - \sqrt{2} \\ &= \sqrt{(2 + \sqrt{2})^2} - \sqrt{2} \\ &= |2 + \sqrt{2}| - \sqrt{2} \\ &= 2 + \sqrt{2} - \sqrt{2} \\ &= 2. \end{aligned}$$

Thus, both are rational as $1, 2 \in \mathbb{Q}$. $\square$

Problem 6 (Ross 3.6)**.** *We prove the triangle inequality for any n real numbers.*

6.a. *For any $a, b, c \in \mathbb{R}$, $|a + b + c| \leq |a| + |b| + |c|$.*

Proof. Observe:

$$|a + b + c| = |a + (b + c)| \leq |a| + |b + c| \leq |a| + |b| + |c|.$$

$\square$

6.b. *For any $a_1, a_2, \ldots, a_n \in \mathbb{R}$,*

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|.$$

Proof. We proceed by induction.

When $n = 0$, observe that

$$\left|\sum_{i=1}^{0} a_i\right| = 0 \leq 0 = \sum_{i=1}^{n} |a_i|,$$

Now, assume that the triangle inequality holds for n real numbers. We show it holds for $n+1$. We have:

$$\begin{aligned}\left|\sum_{i=1}^{n+1} a_i\right| &= \left|\sum_{i=1}^{n} a_i + a_{n+1}\right| \\ &\leq \left|\sum_{i=1}^{n} a_i\right| + |a_{n+1}| \\ &\leq \sum_{i=1}^{n} |a_i| + |a_{n+1}| \\ &\leq \sum_{i=1}^{n+1} |a_i|.\end{aligned}$$

□

Problem 7 (Ross 4.11). *For any $a, b \in \mathbb{R}$ with $a < b$, there exists an infinite number of rational numbers between a and b.*

Proof. In order to show this statement, we will show for any positive integer n, there exists n rational numbers strictly between a and b. Thus, for any k, there cannot be exactly k rational numbers between a and b as there exists $k+1$ such numbers, showing that there are infinitly many rational numbers.

By the denseness of $\mathbb{R}$, there exists a $p \in \mathbb{Q}$ such that $a < p < b$. Similarly, as $p < b$ and $p, b \in \mathbb{R}$, there exists a $q \in \mathbb{Q}$ such that $p < q < b$. Now consider the numbers $r_0, r_1, \ldots, r_n \in \mathbb{R}$ such that

$$r_i = \frac{(n-i)p + iq}{n}.$$

Observe that $r_i \in \mathbb{Q}$ as $n, i, p, q \in \mathbb{Q}$ and $\mathbb{Q}$ is a field. We claim that $p < r_i < q$. Notice that:

$$\begin{aligned} p &< q \\ \frac{ip}{n} &< \frac{iq}{n} \\ p &< \frac{(n-i)p + iq}{n}. \end{aligned}$$

Similarly:

$$\begin{aligned} p &< q \\ \frac{(n-i)p}{n} &< \frac{(n-i)q}{n} \\ \frac{(n-i)p + iq}{n} &< q. \end{aligned}$$

Thus, $p < r_i < q$ meaning $a < r_i < b$ as $a < p < q < b$. Thus, as $r_1, r_2, \ldots, r_n$ are rational numbers strictly between a and b, there are at least n rational numbers strictly between a and b. This proves the proposition. □

Problem 8 (Ross 4.14). *For any $A, B \subseteq \mathbb{R}$, we define*

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

8.a. *For any nonempty $A, B \subseteq \mathbb{R}$, $A+B$ is bounded above if and only if A is bounded above and B is bounded above, and* $\sup(A + B) = \sup A + \sup B$*.*

Proof. First we show that $A+B$ is bounded above iff A and B are bounded above. Assume now that $A+B$ has upper bound r. We will show that A is bounded above; it follows similarly that B is bounded above. Since B is nonempty, there exists a $b \in B$. Now for any $a \in A$, observe that $a + b \in A + B$, so $a + b \leq r$. Thus, $a \leq r - b$, meaning that $r - b$ is an upper bound for A. Thus, A and B are bounded above. Conversely, if A has upper bound r and B has upper bound s, then for all $a + b \in A + B$, $a + b \leq r + s$ meaning that $r + s$ is an upper bound for $A + B$.

Now we show that $\sup(A+B) = \sup A + \sup B$, assuming either $A+B$ is bounded above or equivalently, A and B are bounded above. For any $a + b \in A + B$, observe that $a \leq \sup A$ and $b \leq \sup B$, so $a + b \leq \sup A + \sup B$, meaning that $\sup A + \sup B$ is an upper bound on $A+B$ and thus $\sup(A+B) \leq \sup A + \sup B$. Now, take any $b \in B$. For any $a \in A$, observe that $a + b \in A + B$, so $a + b \leq \sup(A + B)$. Thus, $a \leq \sup(A + B) - b$, so $\sup(A + B) - b$ is an upper bound for A and thus $\sup A \leq \sup(A + B) - b$. Rearranging, $b \leq \sup(A+B) - \sup A$; therefore, $\sup B \leq \sup(A+B) \sup A$. So, $\sup A + \sup B \leq \sup(A+B)$. Because of the antisymmetry of $\leq$, $\sup A + \sup B = \sup(A + B)$. □

8.b. *For any nonempty $A, B \subseteq \mathbb{R}$, $A+B$ is bounded below if and only if A is bounded below and B is bounded below. In either case,* $\inf(A + B) = \inf A + \inf B$*.*

Proof. We first show $A + B$ is bounded below iff A is bounded below and B is bounded below. Suppose now that $A + B$ has lower bound r. As B is nonempty, there exists a $b \in B$. For all $a \in A$, observe that $a + b \in A + B$, so $r \leq a + b$ and thus $r - b \leq a$, implying that $r - b$ is a lower bound of A. It similarly follows that B is bounded below. Conversely, assume that A and B are bounded below. Let r be a lower bound of A and s a lower bound of B. For any $a + b \in A + B$ with $a \in A$ and $b \in B$, we have $r \leq a$ and $s \leq b$, so $r + s \leq a + b$, implying $r + s$ is a lower bound of $A + B$. Thus, $A + B$ is bounded below.

Assuming that both sets are bounded below, we show $\inf(A + B) = \inf A + \inf B$. For any $a + b \in A + B$ with $a \in A$ and $b \in B$, notice that $\inf A \leq a$ and $\inf B \leq b$, so $\inf A + \inf B \leq a + b$. Thus, $\inf A + \inf B \leq \inf(A + B)$ as it is a lower bound on $A + B$. For any $b \in B$, observe that for all $a \in A$, $a + b \in A + B$, so $\inf(A+B) \leq a+b$ and $\inf(A+B)-b \leq a$. Thus, $\inf(A+B)-b$ is a lower bound of A, so $\inf(A+B)-b \leq \inf A$. Thus, $\inf(A + B) - \inf A \leq b$ for all $b \in B$ meaning that $\inf(A + B) - \inf A$ is a lower bound of B and $\inf(A + B) - \inf A \leq \inf B$. Thus, $\inf(A + B) \leq \inf A + \inf B$. By the antisymmetry of $\leq$, we thus have $\inf(A + B) = \inf A + \inf B$. □

Problem 9 (Ross 7.5)

Problem 9.a Observe that

$$s_n = \sqrt{n^2 + 1} - n = \frac{(\sqrt{n^2 + 1} - n)(\sqrt{n^2 + 1} + n)}{\sqrt{n^2 + 1} + n} = \frac{(n^2 + 1) - n^2}{\sqrt{n^2 + 1} + n} = \frac{1}{\sqrt{n^2 + 1} + n}.$$

So, $\lim s_n = 0$ as the denominator goes to $+\infty$.

Problem 9.b As before, we have

$$\sqrt{n^2 + n} - n = \frac{(\sqrt{n^2 + n} - n)(\sqrt{n^2 + n} + n)}{\sqrt{n^2 + n} + n} = \frac{n^2 + n - n^2}{\sqrt{n^2 + n} + n} = \frac{n}{\sqrt{n^2 + n} + n} = \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}.$$

As the limit of the denominator is $\sqrt{1} + 1 = 2$, the limit of this sequence is $\frac{1}{2}$.

Problem 9.c We have

$$\begin{aligned}\sqrt{4n^2+n}-2n &= \frac{(\sqrt{4n^2+n}-2n)(\sqrt{4n^2+n}+2n)}{\sqrt{4n^2+n}+2n}\\ &= \frac{4n^2+n-4n^2}{\sqrt{4n^2+n}+2n}\\ &= \frac{n}{\sqrt{4n^2+n}+2n}\\ &= \frac{1}{\sqrt{4+\frac{1}{n}}+2}.\end{aligned}$$

Observe that the limit of the denominator is $\sqrt{4}+2=4$ as the limit of $\frac{1}{n}$ is 0. So, the limit of this sequence is $\frac{1}{4}$.