104 Set 10

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Ross 33.4. Consider the function $f : [0, 1] \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then observe that $|f(x)| = 1$ for all $x \in [0, 1]$, meaning that $|f|$ is integrable. However, f is not integrable, since we have:

$$\frac{1}{2}(f(x) + 1) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

This function is not integrable, as we have shown. If f is integrable, then this function is also integrable, which is impossible. Hence, f is integrable, though $|f| = 1$ is not.

Ross 33.7.a. We start by showing that for any bounded function $f : [a, b] \to \mathbb{R}$ such that $|f(x)| \leq B$ for all $x \in [a, b]$, the following inequality holds:

$$\sup_{x\in[a,b]} f^2(x) - \inf_{y\in[a,b]} f^2(y) \leq 2B \left(\sup_{x\in[a,b]} f(x) - \inf_{y\in[a,b]} f(y) \right)$$

We have:

$$\begin{aligned} |f^2(x) - f^2(y)| &= |(f(x) - f(y))(f(x) + f(y))| \\ &= |f(x) - f(y)||f(x) + f(y)| \\ &\leq |f(x) - f(y)|(|f(x)| + |f(y)|) \\ &\leq 2B|f(x) - f(y)|. \end{aligned}$$

Let $m = \inf_{x\in[a,b]} f(x)$ and $M = \sup_{x\in[a,b]} f(x)$. If $f(x) \geq f(y)$, then $|f(x) - f(y)| = f(x) - f(y) \leq M - m$. If $f(x) < f(y)$, then $|f(x) - f(y)| = f(y) - f(x) \leq M - m$. Hence, $|f(x) - f(y)| \leq M - m$, and so:

$$2B(M - m) \geq |f^2(x) - f^2(y)| \geq f^2(x) - f^2(y).$$

Taking the sup with respect to x, we have:

$$\sup_{x\in[a,b]} f^2(x) - f^2(y) \leq 2B(M - m).$$

Therefore,

$$\sup_{x\in[a,b]} f^2(x) - 2B(M - m) \leq f^2(y).$$

meaning

$$\sup_{x\in[a,b]} f^2(x) - 2B(M - m) \leq \inf_{y\in[a,b]} f^2(y).$$

Hence, we have our inequality:

$$\sup_{x\in[a,b]} f^2(x) - \inf_{y\in[a,b]} f^2(y) \leq 2B(M - m) = 2B \left(\sup_{x\in[a,b]} f(x) - \inf_{y\in[a,b]} f(y) \right).$$

Now, consider any bounded function $f : [a, b] \to \mathbb{R}$, and a partition $P = \{p_0, p_1, \cdots, p_n\}$. Then we have:

$$\begin{aligned} U(f^2, P) - L(f^2, P) &= \sum_{i=1}^{n} \left(\sup_{x \in [p_{i-1}, p_i]} f^2(x) - \inf_{x \in [p_{i-1}, p_i]} f^2(x) \right) (p_i - p_{i-1}) \\ &\leq \sum_{i=1}^{n} 2B \left(\sup_{x \in [p_{i-1}, p_i]} f(x) - \inf_{x \in [p_{i-1}, p_i]} f(x) \right) (p_i - p_{i-1}) \\ &= 2B(U(f, P) - L(f, P)). \end{aligned}$$

Ross 33.7.b. Let $f : [a, b] \to \mathbb{R}$ be integrable with $|f| \leq B$. Take any $\epsilon > 0$. Then there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \frac{\epsilon}{2B}.$$

Thus,

$$U(f^2, P) - L(f^2, P) \leq 2B(U(f, P) - L(f, P)) < \epsilon$$

and so f^2 is integrable.

Ross 33.13. We use the Intermediate Value Theorem for integrals (Ross Thm. 33.9, which can be proven using the Intermediate Value Theorem for continuous functions). Since $\int_a^b f = \int_a^b g$, $\int_a^b (f - g) = 0$. Thus, there exists an $x \in (a, b)$ such that

$$f(x) - g(x) = \frac{1}{b - a} \int_a^b (f - g) = 0$$

and so $f(x) = g(x)$.

Ross 35.4. We compute these integrals using Integration by Parts, Ross Thm 35.19. We have:

$$\int_a^b x dF(x) + \int_a^b F(x) dx = bF(b) - aF(a)$$

since both x and $F(x) = \sin t$ are continuous. In particular, this gives:

$$\int_a^b x dF(x) = b \sin b - a \sin a - \int_a^b \sin x dx = b \sin b + \cos b - a \sin a - \cos a$$

using the Fundamental Theorem of Calculus.

a. We have:

$$\int_0^{\frac{\pi}{2}} x dF(x) = \frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2} = \frac{\pi}{2}.$$

b. We have:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2} - \left(-\frac{\pi}{2} \sin \left(-\frac{\pi}{2} \right) - \cos \left(-\frac{\pi}{2} \right) \right) = \pi.$$

Ross 35.9.a. Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and $F : [a, b] \to \mathbb{R}$ increasing. Then there exists some $x, y \in [a, b]$ such that $f(x)$ is the maximum value of f, and $f(y)$ is the minimum value of f, by the Extreme Value Theorem. We can use these to bound the upper and lower sums. For any partition $P = \{p_0, p_1, \ldots, p_n\}$ where $a = p_0 < p_1 < \cdots < p_n = b$, we have:

$$\begin{aligned} L(P, f, F) &= \sum_{i=1}^{n} \left(\inf_{t \in [p_{i-1}, p_i]} f(t) \right) (F(p_i) - F(p_{i-1})) \\ &\geq \sum_{i=1}^{n} f(y)(F(p_i) - F(p_{i-1})) \\ &= f(y)(F(p_n) - F(p_0)) \\ &= f(y)(F(b) - F(a)). \end{aligned}$$

We similarly have $U(P, f, F) \leq f(x)(F(b) - F(a))$. Hence, we have

$$f(y)(F(b) - F(a)) \leq \int_a^b f dF \leq f(x)(F(b) - F(a))$$

which means

$$f(y) \leq \frac{1}{F(b) - F(a)} \int_a^b f dF \leq f(x).$$

We can now apply the Intermediate Value Theorem on the interval $[x, y]$ or $[y, x]$, depending on if $x \leq y$ or $y \leq x$ to see that there exists some $c \in [x, y]$ with

$$f(c) = \frac{1}{F(b) - F(a)} \int_a^b f dF.$$

Thus, we have $c \in [a, b]$ and

$$f(c)(F(b) - F(a)) = \int_a^b f dF.$$