104 Set 11

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34.2.a. Let $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = \int_0^x e^{t^2} dt$. We wish to compute $\lim_{x\to 0} \frac{F(x)}{x}$. Note that $F(0) = \int_0^0 e^{t^2} dt = 0$. Thus,

$$\lim_{x\to 0} \frac{F(x)}{x} = \lim_{x\to 0} \frac{F(x) - F(0)}{x} = F'(0).$$

By the Fundamental Theorem of Calculus, as e^{t^2} is continuous, $F'(x) = e^{x^2}$. Hence, $F'(0) = e^0 = 1$, and so $\lim_{x\to 0} \frac{1}{x} \int_0^x e^{t^2} dt = 1$.

34.2.b. We similarly approach finding $\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}$. Let $F(x) = \int_3^x e^{t^2} dt$. Noting $0 = F(3)$, we have:

$$\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2} dt = \lim_{h\to 0} \frac{F(3+h)}{h} = \lim_{h\to 0} \frac{F(3+h) - F(3)}{h} = F'(3).$$

As before, as e^{t^2} is continuous, by the Fundamental Theorem of Calculus, $F'(x) = e^{x^2}$. Hence, $F'(3) = e^9$ and so $\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2} dt = e^9$.

34.5. We have

$$F(x) = \int_{x-1}^{x+1} f(t)dt = \int_1^{x+1} f(t)dt + \int_{x-1}^{1} f(t)dt = \int_1^{x+1} f(t)dt - \int_1^{x-1} f(t)dt.$$

The First Fundamental Theorem of Calculus (with the derivative chain rule) tells us each of these integrals are differentiable. In particular:

$$\frac{d}{dx} \int_1^{x+1} f(t)dt = f(x+1) \frac{d}{dx}(x+1) = f(x+1)$$
$$\frac{d}{dx} \int_1^{x-1} f(t)dt = f(x-1) \frac{d}{dx}(x-1) = f(x-1).$$

Therefore, as the sum of differentiable functions, F is differentiable, and $F'(x) = f(x+1) - f(x-1)$.

34.7. Let $I = \int_0^1 x\sqrt{1-x^2}dx$. We use the change of variables theorem (Ross Theorem 34.4) to solve this problem. Let $u(x) = 1 - x^2$, and observe that $u'(x) = -2x$. Hence, we have

$$I = \int_0^1 x\sqrt{1-x^2}dx = -\frac{1}{2} \int_0^1 \sqrt{u}u'(x)dx.$$

Now, as the function $x \mapsto \sqrt{x}$ is continuous, we have by the Change of Variables Theorem:

$$I = -\frac{1}{2} \int_{u(0)}^{u(1)} \sqrt{x}dx.$$

Note that $u(0) = 1$ and $u(1) = 0$. Also, $\sqrt{x}$ is continuous on this interval and, by the Power Rule, has antiderivative $\frac{2}{3}x^{\frac{3}{2}}$. Hence, we have

$$I = -\frac{1}{2} \int_1^0 \sqrt{x}dx = -\frac{1}{2} \left(\frac{2}{3}0^{\frac{3}{2}} - \frac{2}{3}1^{\frac{3}{2}} \right) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}.$$