104 Set 2

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Ross 9.9. Let s and t be sequences in $\mathbb{R}$ and $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $s_n \leq t_n$.

a. If $\lim_{n\to\infty} s_n = +\infty$ then $\lim_{n\to\infty} t_n = +\infty$.

Proof. For all $M \in \mathbb{R}$, there exists some $N \in \mathbb{N}$ such that if $n \geq N$, then $s_n > M$. So, for all $n \geq \max(N, N_0)$, $t_n \geq s_n > M$. Thus, $\lim_{n\to\infty} s_n = +\infty$. □

b. If $\lim_{n\to\infty} t_n = -\infty$ then $\lim_{n\to\infty} s_n = -\infty$.

Proof. For all $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that $t_n < M$ whenever $n \geq N$. So, for all $n \geq \max(N, N_0)$, $s_n \leq t_n < M$. Thus, $\lim_{n\to\infty} s_n = -\infty$. □

c. If $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, then $s \leq t$.

Proof. There are a few good ways to prove this; one way is to show that the limit of a sequence of nonnegative real numbers is nonnegative, or alternatively, we could consider the $\liminf$ of s_n and $\limsup$ of t_n, observing they are s and t respectively as both sequences converge, and as $s_n \leq t_n$ for some natural number n, this implies $s \leq t$.

For this problem, however, we will do a proof by contradiction. Suppose that $s > t$. Then there exists some $\epsilon < \frac{s-t}{2}$. Now, there exists some $N \in \mathbb{N}$ such that if $n \geq N$ then $|s_n - s| < \epsilon$ or equivalently, $s - \epsilon < s_n < s + \epsilon$. Likewise, there exists some $M \in \mathbb{N}$ such that if $n \geq M$, then $t - \epsilon < t_n < t + \epsilon$. Let $n \geq N, M$, and observe that $s - \epsilon < s_n$ and $t_n < t + \epsilon$. We have

$$s - \epsilon > s - \frac{s-t}{2} = \frac{s+t}{2}$$

and

$$t + \epsilon < t - \frac{s-t}{2} = \frac{s+t}{2}.$$

Thus,

$$t_n < t + \epsilon < \frac{s+t}{2} < s - \epsilon < s_n.$$

But, by hypothesis, $s_n \leq t_n$. So, this is impossible and $s \leq t$. □

Ross 9.15. For all $a \in \mathbb{R}$, $\lim_{n\to\infty} \frac{a^n}{n!} = 0$.

Proof. By the Archimedian Principle, there exists some positive integer $k > |a|$. Observe that $\left|\frac{a^n}{n!}\right| = \frac{|a|^n}{n!} < \frac{k^n}{n!}$ as $|a| \geq 0$. For $n > k$, notice:

$$\frac{k^n}{n!} = \frac{k^k}{k!} \cdot \prod_{i=k+1}^{n} \left(\frac{k}{i}\right) \leq \frac{k^k}{k!} \cdot \left(\frac{k}{k+1}\right)^{n-k}$$

as for all $i \geq k$, $\frac{k}{k+1} \leq \frac{k}{i}$. Observe that

$$\lim_{n\to\infty} \left(\frac{k}{k+1}\right)^n = 0$$

and so

$$\lim_{n\to\infty} \frac{k^k}{k!}\left(\frac{k}{k+1}\right)^n = 0.$$

Thus, for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$\left|\frac{k^k}{k!}\left(\frac{k}{k+1}\right)^n\right| < \epsilon.$$

Now, if $n \geq N + k$ (observing that $N + k \in \mathbb{N}$), then $n - k \geq N$, so

$$\epsilon > \left|\frac{k^k}{k!}\left(\frac{k}{k+1}\right)^{n-k}\right| = \frac{k^k}{k!}\left(\frac{k}{k+1}\right)^{n-k} \geq \frac{k^n}{n!} \geq \frac{|a|^n}{n!} = \left|\frac{a^n}{n!}\right|.$$

So, $\lim_{n\to\infty} \frac{a^n}{n!} = 0$. □

Ross 10.7 For any $S \subseteq \mathbb{R}$ bounded above with $\sup S \notin S$, there exists a sequence a such that for all $n \in \mathbb{N}$, $a_n \in S$ and $\lim_{n\to\infty} a_n = \sup S$.

Proof. By definition of supremum, for any $\epsilon > 0$, there exists an $s \in S$ such that $\sup S - s < \epsilon$. For all positive integers n, let $s_n \in S$ such that $\sup S - s_n < \frac{1}{n}$. We claim that $\lim_{n\to\infty} s_n = \sup S$. For every $\epsilon > 0$, there exists some integer $N > \frac{1}{\epsilon}$, so for all $n \geq N$, $n > \frac{1}{\epsilon}$ and $\epsilon > \frac{1}{n} > \sup S - s_n = |s_n - \sup S|$. Thus, $\lim_{n\to\infty} s_n = \sup S$. □

Ross 10.8 For any increasing sequence s, the sequence σ, where

$$\sigma_n = \frac{1}{n}\sum_{i=1}^{n} s_i$$

is increasing.

Proof. First, notice for all n,

$$\sigma_n = \frac{1}{n}\sum_{i=1}^{n} s_i \leq \frac{1}{n}\sum_{i=1}^{n} s_n = s_n.$$

Now, for all $n > 1$,

$$\begin{aligned}
n\sigma_n &= \sum_{i=1}^{n} s_i \\
&= \sum_{i=1}^{n-1} s_i + s_n \\
&= (n-1)\sigma_{n-1} + s_n \\
&\geq (n-1)\sigma_{n-1} + s_{n-1} \\
&\geq (n-1)\sigma_{n-1} + \sigma_{n-1} \\
&= \sigma_{n-1}.
\end{aligned}$$

So, σ is also increasing. □

Ross 10.9 We have $s_1 = 1$ and $s_{n+1} = \frac{ns_n^2}{n+1}$ for $n \geq 1$.

a. We have $s_2 = \frac{1 \cdot s_1^2}{2} = \frac{1}{2}$, $s_3 = \frac{2}{3} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{6}$, and

$$s_4 = \frac{3}{4} \cdot \left(\frac{1}{6}\right)^2 = \frac{1}{48}.$$

b. We show by induction that $0 < s_n \leq 1$. Observe that $s_1 \leq 1$. For all n, if $s_n \leq 1$, then

$$s_{n+1} = \frac{ns_n^2}{n+1} \leq s_n^2 \leq 1.$$

So, $s_n \leq 1$. As $\frac{n}{n+1} > 0$ and $s_n^2 > 0$, $s_{n+1} > 0$ as well. Thus, by induction, $0 < s_n < 1$ for all n.

Now, this implies $s_{n+1} \leq s_n^2$ and as $s_n < 1$, $s_n^2 \leq s_n$; thus, s_n is decreasing. Also, since $s_n > 0$ for all n, s_n is bounded below and so convergent.

c. Let $s = \lim_{n\to\infty} s_n$. Then we have $s = \lim_{n\to\infty} s_{n+1}$. Thus:

$$s = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \left(\frac{ns_n^2}{n+1}\right) = \lim_{n\to\infty} \frac{n}{n+1} \lim_{n\to\infty} s_n^2 = \lim_{n\to\infty} \frac{n}{n+1} \left(\lim_{n\to\infty} s_n\right)^2 = 1 \cdot s^2 = s^2.$$

So, as $s^2 = s$, $s(s-1) = 0$, so either $s = 0$ or $s = 1$. Since s_n is decreasing, s is the infimum of the sequence. Thus, as $s_1 = \frac{1}{2} < 1$, $s \neq 1$, and therefore $s = 0$.

Ross 10.10 We have $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$.

a. We have $s_2 = \frac{1}{3}(1+1) = \frac{2}{3}$, $s_3 = \frac{1}{3}(\frac{2}{3}+1) = \frac{5}{9}$ and $s_4 = \frac{1}{3}(\frac{5}{9}+1) = \frac{14}{27}$.

b. Observe that $s_1 > \frac{1}{2}$. Now, if $s_n > \frac{1}{2}$, then $s_{n+1} = \frac{1}{3}(s_n+1) > \frac{1}{3}(\frac{1}{2}+1) = \frac{1}{2}$. Thus, by induction $s_n > \frac{1}{2}$ for all n.

c. Since $s_n > \frac{1}{2}$, $2s_n > 1$, so $s_{n+1} = \frac{1}{3}(s_n+1) < \frac{1}{3}(s_n + 2s_n) = s_n$. Thus, s_n is a decreasing sequence.

d. It follows, as s_n has lower bound $\frac{1}{2}$, that s_n is convergent. Let $s = \lim_{n\to\infty} s_n$. So,

$$s = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} (\frac{1}{3}(s_n+1)) = \frac{1}{3}\left(\lim_{n\to\infty}(s_n) + 1\right) = \frac{1}{3}(s+1).$$

Thus, $3s = s+1$ and so $2s = 1$, implying $s = \frac{1}{2}$.

Ross 10.11 We have $t_1 = 1$ and $t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n$.

a. For all n, $\frac{1}{4n^2} > 0$, so $1 - \frac{1}{4n^2} < 1$, meaning that $t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n < t_n$.

b. 0, perhaps?

Theorem (Squeeze Theorem)**.** *Let a_n, b_n, and c_n be sequences such that* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = b$ *and for all n, $a_n \leq b_n \leq c_n$. Then* $\lim_{n\to\infty} b_n = b$.

Proof. For any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_n - b| < \epsilon$ whenever $n \geq N$. Also, there exists an $M \in \mathbb{N}$ such that $|c_n - b| < \epsilon$ whenever $n \geq M$. For any $n \geq \max(N, M)$, observe both of these are true. So, $b - \epsilon < a_n$ and $c_n < b + \epsilon$. As $a_n \leq b_n \leq c_n$, this means $b - \epsilon < b_n < b + \epsilon$. Thus, $|b_n - b| < \epsilon$. Hence, $\lim_{n\to\infty} b_n = b$. □