104 Set 3

Ishaan Patkar

10.6.a. If (s_n) is a sequence such that for all $n \in \mathbb{N}$, $|s_{n+1} - s_n| < 2^{-n}$ then (s_n) is a Cauchy sequence.

Proof. For any $n, m \in \mathbb{N}$, assuming $m > n$, we have:

$$|s_m - s_n| = \left| \sum_{i=n+1}^{m} (s_i - s_{i-1}) \right| \leq \sum_{i=n+1}^{m+n} |s_i - s_{i-1}| \leq \sum_{i=n+1}^{m+n} 2^{-i} = \frac{2^{-n-1} - 2^{-m-n-1}}{1 - 2^{-1}} = \frac{1}{2^n} - \frac{1}{2^{n+m}} < \frac{1}{2^n}.$$

So, for any $n, m \in \mathbb{N}$, we have:

$$|s_m - s_n| < \frac{1}{2^{\min(m,n)}} = \max\left(\frac{1}{2^n}, \frac{1}{2^m}\right),$$

since if $m > n$, then $|s_m - s_n| < \frac{1}{2^n}$, and if $n > m$, then $|s_m - s_n| < \frac{1}{2^m}$ and if $m = n$ then $|s_m - s_n| = 0$ and so satisfies both inequalities.

Now, as $\lim_{n\to\infty} \frac{1}{2^n} = 0$ as $\left|\frac{1}{2}\right| < 1$, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\frac{1}{2^n} = \left|\frac{1}{2^n}\right| < \epsilon$ for all $n > N$. So, for all $m, n > N$:

$$|s_m - s_n| < \min\left(\frac{1}{2^m}, \frac{1}{2^n}\right) < \epsilon.$$

Hence, (s_n) is Cauchy. □

10.6.b. This statement would actually imply that the harmonic series is convergent. Let:

$$s_n = \sum_{i=1}^{n} \frac{1}{i} = s_{n-1} + \frac{1}{n}.$$

Observe that for all $n \in \mathbb{N}$, $|s_{n+1} - s_n| = \frac{1}{n+1} < \frac{1}{n}$. Assuming this statement is true, then (s_n) is Cauchy and so convergent. But this implies that the harmonic series is convergent, which it is not.

11.2.a. For (a_n), consider the sequence $(a_{2n})_{n=1}^{\infty}$. Observe that $a_{2n} = 1$ so this sequence is monotonic. For (b_n), the sequence itself is monotonic, so any subsequence is monotonic and will do. Likewise for (c_n); it is also monotonic and any subsequence is monotonic. For (d_n), observe that:

$$d_n = \frac{6n + 4}{7n - 3} = \frac{6}{7} + \frac{46}{7} \cdot \frac{1}{7n - 3}.$$

As $\frac{1}{7n-3}$ is monotonically decreasing and has positive coefficient in this sequence, (d_n) is also monotonically decreasing and any subsequence is thus monotonic.

11.2.b. Clearly, 1 and -1 are subsequential limits of (a_n) since there is a subsequence containing only 1's and a subsequence containing only -1's. In fact, the lim sup is 1 and the lim inf is -1 as the supremum and infimum of all $n > N$ for any $N \in \mathbb{N}$ is 1 and -1 respectively. We desire to show these are the only subsequential limits of (a_n). To see this, consider any subsequential limit a. Since it is a subsequential limit, it must be between -1 and 1. Suppose it is neither 1 nor -1. Then $-1 < a < 1$, or $|a| < 1$. Let ϵ such that $0 < \epsilon < \min(a + 1, 1 - a)$. So $\epsilon < a + 1$, implying $-1 < a - \epsilon$, and as $\epsilon < 1 - a$, $a + \epsilon < 1$. Since a is a subsequential limit of (a_n), there exists infinitely many $n \in \mathbb{N}$ such that $a_n \in (a - \epsilon, a + \epsilon)$. However, either $a_n = 1$ or $a_n = -1$, which would respectively imply that $1 < 1$ and $-1 < -1$, either of which are impossible. Hence we have a contradiction, so either $a = 1$ or $a = -1$.

The other sequences are convergent (in the extended real numbers) and so their sets of subsequential limits are easier to define. Note $\lim_{n\to\infty} b_n = 0$, so 0 is the only subsequential limit. We have $\lim_{n\to\infty} c_n = +\infty$. Also, $\lim_{n\to\infty} d_n = \frac{6}{7}$ by the decomposition in the previous part and as $\frac{1}{7n-3} \to 0$ as $n \to \infty$.

11.2.c. These can be determined from the previous part, being the maximum and the minimum of the set of subsequential limits, respectively.

$$\begin{array}{cc} \limsup_{n\to\infty} a_n = 1 & \liminf_{n\to\infty} a_n = -1 \\ \limsup_{n\to\infty} b_n = 0 & \liminf_{n\to\infty} b_n = 0 \\ \limsup_{n\to\infty} c_n = +\infty & \liminf_{n\to\infty} c_n = +\infty \\ \limsup_{n\to\infty} d_n = \frac{6}{7} & \liminf_{n\to\infty} d_n = \frac{6}{7} \end{array}$$

11.2.d. It follows from part (b) that (a_n) diverges, (b_n) converges to 0, (c_n) diverges to $+\infty$, and (d_n) converges to $\frac{6}{7}$.

11.2.e. Observe that $\sup\{a_n\}_{n=1}^{\infty} = 1$ and $\inf\{a_n\}_{n=1}^{\infty} = -1$ as 1 and -1 are the maximum and minimum of the set of values the sequence takes, respectively. Thus, (a_n) is bounded. As (b_n) converges, it is bounded. As (c_n) diverges to $+\infty$, it is not bounded above, though as $n^2 \geq 0$ for all $n \in \mathbb{N}$, it is bounded below. As (d_n) converges, it is bounded.

11.3.a. For (s_n), observe that:

$$s_n = \cos\left(\frac{n\pi}{3}\right) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod 3 \\ \frac{1}{2} & \text{if } x \equiv 1 \pmod 3 \\ -\frac{1}{2} & \text{if } x \equiv 2 \pmod 3. \end{cases}$$

One monotone subsequence is $(s_{3n})_{n=1}^{\infty}$, as $s_{3n} = 0$.

For (t_n), it is monotone as

$$t_n = \frac{3}{4n+1} > \frac{3}{4n+5} = t_{n+1}.$$

For (u_n), we have $(u_{2n})_{n=1}^{\infty}$ monotone as

$$u_{2n} = \frac{1}{2^{2n}} > \frac{1}{2^{2n+2}} = u_{2(n+1)}.$$

For (v_n), we have $(v_{2n})_{n=1}^{\infty}$ also monotone as

$$v_{2n} = 1 + \frac{1}{2n} > 1 + \frac{1}{2n+2} = v_{2(n+1)}.$$

11.3.b. The set of subsequential limits of (s_n) is $\{-\frac{1}{2}, 0, \frac{1}{2}\}$; for (t_n) is $\{0\}$ as it converges to 0; for (v_n) is $\{0\}$ as (v_n) converges to 0 since it is a geometric sequence with common ratio between -1 and 1; and for (u_n) is $\{-1, 1\}$. The proofs of the subsequential limits for (s_n) and (v_n) follow similarly to the proof for (a_n) in the first problem and will be omitted (since the question doesn't ask for a proof). We simply justify these sets by noting that (s_n) only takes on a finite number of values, and finite sets are closed, and that v_n only gets arbitrarily close to 1 and -1.

11.3.c. Noting that the lim sup and lim inf are the maximum and minimum in the set of subsequential limits, we have the following:

$$\begin{array}{cc} \limsup_{n\to\infty} s_n = \frac{1}{2} & \liminf_{n\to\infty} s_n = -\frac{1}{2} \\ \limsup_{n\to\infty} t_n = 0 & \liminf_{n\to\infty} t_n = 0 \\ \limsup_{n\to\infty} u_n = 0 & \liminf_{n\to\infty} u_n = 0 \\ \limsup_{n\to\infty} v_n = 1 & \liminf_{n\to\infty} v_n = -1 \end{array}$$

11.3.d. By the set of subsequential limits, the sequence (s_n) diverges, (t_n) converges to 0, (u_n) converges to 0, and (v_n) diverges.

11.3.e. All four sequences are bounded. Observe that $-\frac{1}{2} \leq s_n \leq \frac{1}{2}$, (t_n) and (u_n) are bounded since they converge, and $-1 \leq v_n \leq 2$, so (v_n) is bounded.

11.5.a. We claim that the set of subsequential limits of (q_n) is $[0,1]$. First, observe that every subsequence of (q_n) is contained in $[0,1]$; thus, any subsequential limits will be contained in $[0,1]$. Now, for any $x \in [0,1]$ and $\epsilon > 0$, there exists infinitely many rationals p strictly between $\max(0, x-\epsilon)$ and $\min(1, x+\epsilon)$, since $0, x-\epsilon < 1, x+\epsilon$. Since each of these rationals lie in $(0,1]$, they will appear in the sequence (q_n) as it enumerates all of the rationals in $(0,1]$. Each of these infinitely many rationals q_n will also lie between $x-\epsilon$ and $x+\epsilon$, meaning that $\{n \in \mathbb{N} \mid |q_n - x| < \epsilon\}$ is infinite. Thus, x is a subsequential limit of (q_n).

11.5.b. We have $\limsup q_n = 1$ and $\liminf q_n = 0$ as these are the maximum and minimum of the set of subsequential limits, respectively.

limsup. For any sequence (a_n), we define

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_m \mid m \geq n\}.$$

This is different from sup in many regards. For one, sup takes in subsets of $\mathbb{R}$, whereas $\limsup$ takes in sequences in $\mathbb{R}$. Also, unlike sup, $\limsup$ is also a limit of a particular type of sequence generated from the sequence (a_n).

Also, $\limsup$ has a number of interesting, counter-intuitive aspects. For instance, the sequence defined by $\limsup$, $\sup\{a_m \mid m \geq n\}$ is decreasing: if we let $A_n = \sup\{a_m \mid m \geq n\}$, then observe for any A_i, A_j with $i \leq j$, for all $m \geq j$, $m \geq j \geq i$, so $a_m \leq A_i$. This means A_i is an upper bound on $\{a_m \mid m \geq j\}$, so, as A_j is the least upper bound, $A_j \leq A_i$. This in some ways counter to the notion that sup represents an upper bound, although it is natural that subsequent upper bounds should be decreasing.

One similarity between $\limsup$ and sup is that $\limsup$ is actually the supremum of a set: specifically, the set of subsequential limits of (a_n) (of which it is also a maximum). In this sense, $\limsup$ is a supremum of limits.