104 Set 4

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Ross 12.10. Let (a_n) be a sequence of real numbers. Then (a_n) is bounded if and only if $\limsup |a_n| < \infty$.

Proof. $\implies$ Since a_n is bounded, there exists an $M \geq 0$ such that $|a_n| \leq M$ for all n. So, as $0 \leq |a_n| \leq M$ for all n, $|a_n|$ is bounded, and thus $\limsup |a_n| < \infty$.

$\impliedby$ Assume that $\limsup |a_n| < \infty$. Let $a = \limsup |a_n|$, and take any $\epsilon > 0$. Then there exists an N such that for all $n \geq N$, $|\sup_{m \geq n} |a_m| - a| < \epsilon$, so $\sup_{m\geq n} |a_m| - a < \epsilon$. So for all $n \geq N$, $|a_n| - a \leq \sup_{m \geq n} |a_m| - a < \epsilon$, so $0 \leq |a_n| < a + \epsilon$. Thus, the set $\{a_n \mid n \geq N\}$ is bounded. Observe $\{a_n \mid n < N\}$ is bounded as well, since it is finite; we can construct bounds by taking the minimum and maximum of the elements in the set. So, the set $\{|a_n| \mid n > 0\}$ is bounded as it is the union of these two bounded sets; we can construct an upper bound by taking the maximum of the upper bound of the two sets, and we can construct a lower bound by taking the minimum of the lower bound of the two sets.

In particular, this means there exists some $M \geq 0$ such that $|a_n| \leq M$ for all n. Therefore, $-M \leq a_n \leq M$ for all n, and so (a_n) is bounded. □

Ross 12.12. Let (s_n) be a sequence of nonnegative reals, and σ_n such that

$$\sigma_n = \frac{1}{n}\sum_{i=1}^{n} s_i.$$

a. Then:

$$\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n.$$

Proof. We will prove that $\limsup \sigma_n \leq \limsup s_n$; observe that $\liminf \sigma_n \leq \limsup \sigma_n$ necessarily. The proof for the first inequality, that $\liminf s_n \leq \liminf \sigma_n$, follows the proof for the last inequality and will be omitted.

First, observe that if $\limsup s_n = \infty$, then clearly $\limsup \sigma_n \leq \infty = \limsup s_n$. So, take the case that $\limsup s_n < \infty$. Let $S = \limsup s_n$. For any $\epsilon > 0$, there exists an N such that if $n \geq N$:

$$0 \leq \sup_{m \geq n} s_m - S < \epsilon$$

so $s_n < S + \epsilon$. So, for all $n > N$:

$$\begin{aligned}
\sigma_n = \frac{1}{n}\sum_{i=1}^{n} s_i &= \frac{1}{n}\sum_{i=1}^{N} s_i + \frac{1}{n}\sum_{i=N+1}^{n} s_i \\
&< \frac{1}{n}\sum_{i=1}^{N} s_i + \frac{1}{n}\sum_{i=N+1}^{n} (S+\epsilon) \\
&= \frac{1}{n}\sum_{i=1}^{N} s_i + \frac{(n-N)(S+\epsilon)}{n} \\
&= \frac{1}{n}\sum_{i=1}^{N} s_i - \frac{N}{n}(S+\epsilon) + S + \epsilon.
\end{aligned}$$

Now, let

$$S_n = \frac{1}{n}\sum_{i=1}^{N} s_i - \frac{N}{n}(S+\epsilon) + S + \epsilon.$$

for all $n > N$, and let $S_n = 0$ otherwise. Observe that:

$$\lim S_n = \lim\left(\frac{1}{n}\sum_{i=1}^{N} s_i - \frac{N}{n}(S+\epsilon) + S + \epsilon\right) = \lim\left(\frac{1}{n}\right)\sum_{i=1}^{N} s_i - N(S+\epsilon)\lim\frac{1}{n} + S + \epsilon = S + \epsilon$$

since we can ignore the first, finite part of the sequence.

Therefore, $\limsup S_n = S + \epsilon$. For all $n \geq N$ and $i \geq n$, observe that $\sigma_i \leq S_i \leq \sup_{j\geq n} S_j$, so $\sup_{i\geq n}\sigma_i \leq \sup_{j\geq n} S_j$. Hence,

$$\limsup \sigma_n \leq \limsup S_n = S + \epsilon$$

as again, we can ignore the first N terms of the sequence, for which the inequalities $\sigma_n \leq S_n$ and $\sup_{i\geq n}\sigma_i \leq \sup_{i\geq n} S_i$ may not hold.

So, for all $\epsilon > 0$, we have that $\limsup \sigma_n \leq S + \epsilon$. It follows that $\limsup \sigma_n \leq S = \limsup s_n$.

The proof for the $\liminf$ inequality follows similarly. $\square$

b. If $\lim s_n$ exists, then $\liminf s_n = \limsup s_n = \lim s_n$, so

$$\lim s_n = \liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n = \lim s_n$$

and so $\liminf \sigma_n = \limsup \sigma_n = \lim s_n$. Thus, σ_n converges an $\lim \sigma_n = \lim s_n$.

c. Let $s_n = 1$ if n is even and $s_n = -1$ if n is odd. Observe that $\limsup s_n = \lim 1 = 1$ and $\liminf s_n = \lim(-1) = -1$ as there exist infinitely many s_n near 1 and -1. However:

$$\sigma_{2n} = \frac{1}{2n}\sum_{i=1}^{2n}(-1)^i = \frac{1}{2n}\sum_{i=1}^{n}((-1)^{2i-1} + (-1)^{2i}) = 0$$

and

$$\begin{aligned}\sigma_{2n+1} &= \frac{1}{2n+1}\sum_{i=1}^{2n+1}(-1)^i \\ &= \frac{1}{2n+1}\sum_{i=1}^{2n}(-1)^i + \frac{1}{2n+1}(-1)^{2n+1} \\ &= \frac{1}{2n+1}\sum_{i=1}^{n}((-1)^{2i-1} + (-1)^{2i}) - \frac{1}{2n+1} \\ &= -\frac{1}{2n+1}.\end{aligned}$$

So, $\limsup \sigma_n = \limsup 0 = 0$ as there are infinitely many σ_n close to 0, and $\liminf \sigma_n = \liminf\left(-\frac{1}{2n+1}\right) = 0$ as well, so $\lim \sigma_n = 0$. So, (σ_n) is convergent, while (s_n) is divergent.

Ross 14.2.

a. The series $\sum \frac{n-1}{n^2}$ cannot converge, for if it does converge, then as $\sum \frac{1}{n^2}$ converges, $\sum \frac{(n-1)+1}{n^2} = \sum \frac{1}{n}$ converges, which is impossible.

b. Observe:

$$\sum_{i=1}^{2n}(-1)^i = \sum_{i=1}^{n}((-1)^{2i} + (-1)^{2i+1}) = \sum_{i=1}^{n} 0 = 0.$$

Therefore,

$$\sum_{i=1}^{2n+1}(-1)^i = \sum_{i=1}^{2n}(-1)^i + (-1)^{2n+1} = -1.$$

So, letting $a_n = \sum_{i=1}^n$, we have

$$a_n = \begin{cases} 0 & \text{if } i \text{ is even} \\ -1 & \text{if } i \text{ is odd.} \end{cases}$$

This sequence clearly diverges; for instance, observe that there are infinitely many elements arbitrarily close to 0 and infinitely many elements arbitrarily close to -1, implying that 0 and -1 are subsequential limits, and as $0 \neq -1$, implying that the sequence does not converge.

c. We have $\frac{3}{n^3} = \frac{3}{n^2}$, so the partial sums of the series $\sum \frac{3}{n^3}$ are 3 times those of $\sum \frac{1}{n^2}$. As $\sum \frac{1}{n^2}$ converges, so does $\sum \frac{3}{n^3}$.

d. Applying the Root Test to this sequence gives $\limsup \frac{n^{\frac{3}{n}}}{3}$. We have shown that $\lim n^{\frac{1}{n}} = 1$, so $\lim n^{\frac{3}{n}} = 1$ and so $\lim \frac{n^{\frac{3}{n}}}{3} = \frac{1}{3} < 1$, and so this sequence converges.

e. Applying the Ratio Test gives

$$\frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}} = \frac{(n+1)^2}{n^2(n+1)} = \frac{n+1}{n^2}.$$

Note this sequence converges to $0 < 1$. Thus, the original series $\sum \frac{n^2}{n!}$ converges.

f. Applying the Root Test to this series $\sum \frac{1}{n^n}$ gives

$$\left(\frac{1}{n^n}\right)^{\frac{1}{n}} = \frac{1}{n}$$

which converges to $0 < 1$, so this series converges.

e. Applying the Root Test gives

$$\left(\frac{n}{2^n}\right)^{\frac{1}{n}} = \frac{n^{\frac{1}{n}}}{2}.$$

Since we have shown $n^{\frac{1}{n}} \to 0$ as $n \to \infty$, this sequence converges to $\frac{1}{2}$ and therefore the series $\sum \frac{n}{2^n}$ converges.

Ross 14.10. Let (a_n) be a sequence such that $a_{2n} = \frac{3^n}{2^n}$ and $a_{2n+1} = \frac{3^n}{2^{n+1}}$ and consider the series $\sum a_n$. We start by applying the Ratio Test on this sequence: observe

$$\frac{a_{2n}}{a_{2n-1}} = \frac{\frac{3^n}{2^n}}{\frac{3^{n-1}}{2^n}} = 3$$

and

$$\frac{a_{2n+1}}{a_{2n}} = \frac{\frac{3^n}{2^{n+1}}}{\frac{3^n}{2^n}} = \frac{1}{2}.$$

So:

$$\frac{a_{n+1}}{a_n} = \begin{cases} \frac{1}{2} & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd.} \end{cases}$$

It is thus clear that $\liminf \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} < 1$ and $\limsup \left|\frac{a_{n+1}}{a_n}\right| = 3 > 1$, so the Ratio Test gives no information.

However, the Root Test gives that the series diverges. Notice that

$$a_{2n}^{\frac{1}{2n}} = \left(\frac{3^n}{2^n}\right)^{\frac{1}{2n}} = \frac{\sqrt{3}}{\sqrt{2}}$$

and

$$a_{2n+1}^{\frac{1}{2n+1}} = \left(\frac{3^n}{2^{n+1}}\right)^{\frac{1}{2n+1}} = \frac{3^{\frac{n}{2n+1}}}{2^{\frac{n+1}{2n+1}}}.$$

As $3^{\frac{n}{2n+1}} = 3^{2n}4n + 2 < 3^{2n+1}4n + 2 = \sqrt{3}$ and $2^{\frac{n+1}{2n+1}} = 2^{\frac{2n+2}{4n+2}} > 2^{\frac{2n+1}{4n+2}} = \sqrt{2}$,

$$a_{2n+1}^{\frac{1}{2n+2}} < \frac{\sqrt{3}}{\sqrt{2}}.$$

So, it follows that $\limsup a_n^{\frac{1}{n}} = \frac{\sqrt{3}}{\sqrt{2}}$ as there are infinitely many $a_n^{\frac{1}{n}}$ equal to this number, an no elements greater than this number, meaning that $\sup_{m\geq n} a_m^{\frac{1}{m}} = \frac{\sqrt{3}}{\sqrt{2}}$ for all n. So, by the Root Test, this sequence diverges as $\sqrt{3} > \sqrt{2}$, so $\frac{\sqrt{3}}{\sqrt{2}} > 1$.

Rudin 3-6.

a. Observe that:

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n}\left(\sqrt{i+1} - \sqrt{i}\right) = \sqrt{n+1} + \sum_{i=2}^{n}\left(-\sqrt{i} + \sqrt{i}\right) - \sqrt{1} = \sqrt{n+1} - 1$$

so the partial sums and the series diverge.

b. Similar to the previous problem, we have:

$$\begin{aligned}\sum_{i=1}^{n} a_i = \sum_{i=1}^{n}\frac{\sqrt{i+1}-\sqrt{i}}{i} &= \sum_{i=1}^{n}\left(\frac{\sqrt{i+1}}{i} - \frac{\sqrt{i}}{i}\right)\\ &= -\frac{\sqrt{1}}{1} + \sum_{i=2}^{n}\left(-\frac{\sqrt{i}}{i} + \frac{\sqrt{i}}{i-1}\right) + \frac{\sqrt{n+1}}{n}\\ &= -1 + \sum_{i=2}^{n}\frac{\sqrt{i}}{i(i-1)} + \frac{\sqrt{n+1}}{n}\\ &= -1 + \sum_{i=2}^{n}\frac{1}{\sqrt{i}(i-1)} + \frac{\sqrt{n+1}}{n}.\end{aligned}$$

Note that $\sqrt{i}(i-1) > \sqrt{i-1}(i-1) = (i-1)^{\frac{3}{2}}$, so

$$\begin{aligned}\sum_{i=1}^{n} a_i &= -1 + \sum_{i=2}^{n}\frac{1}{\sqrt{i}(i-1)} + \frac{\sqrt{n+1}}{n}\\ &< -1 + \sum_{i=2}^{n}\frac{1}{(i-1)^{\frac{3}{2}}} + \frac{\sqrt{n+1}}{n}\\ &= -1 + \sum_{i=1}^{n-1}\frac{1}{i^{\frac{3}{2}}} + \frac{\sqrt{n+1}}{n}.\end{aligned}$$

Note that -1, $\sum_{i=1}^{n-1}\frac{1}{i^{\frac{3}{2}}}$, and $\frac{\sqrt{n+1}}{n}$ converge, so this series also converges.

c. Applying the Root Test gives $\sqrt[n]{n} - 1$ which converges to 0 as $n \to \infty$. As $0 < 1$, this sequence converges.

d. We claim that this series converges if and only if $|z| > 1$. First, if this series converges, $\lim_{n\to\infty}\frac{1}{1+z^n} = 0$. Then $\lim_{n\to\infty}(1 + z^n) = \infty$ so $\lim_{n\to\infty} z^n = \infty$. Note this implies that $\lim_{n\to\infty}|z|^n = \infty$; this will occur only if $|z| > 1$ as it is a geometric progression.

On the other hand, assume that $|z| > 1$. We will show this series converges by applying the Root Test. To do this, we first show that for any $w \in \mathbb{C}$ with $|w| < 1$, $\lim_{n\to\infty}\sqrt[n]{|1 + w^n|} = 1$. Observe

$$1 - |w|^n \leq |1 + w^n| \leq 1 + |w|^n$$

by the Triangle Inequality, implying that, as $1 - |w|^n > 0$,

$$\sqrt[n]{1 - |w|^n} \leq \sqrt[n]{|1 + w^n|} \leq \sqrt[n]{1 + |w|^n}.$$

Now, as $|w|^n < 1$, $0 < 1 - |w|^n \leq 1$, so $1 - |w|^n \leq \sqrt[n]{1 - |w|^n}$. Similarly, as $1 \leq 1 + |w|^n$, $\sqrt[n]{1 + |w|^n} \leq 1 + |w|^n$. Therefore

$$1 - |w|^n \leq \sqrt[n]{|1 + w^n|} \leq 1 + |w|^n.$$

Observe that $\lim_{n\to\infty}(1 - |w|^n) = \lim_{n\to\infty}(1 + |w|^n) = 1$. Therefore, by the Squeeze Theorem,

$$\lim_{n\to\infty} \sqrt[n]{|1 + w^n|} = 1.$$

Now, when $|z| > 1$, observe that $\left|\frac{1}{z}\right| < 1$, so $\lim_{n\to\infty} \sqrt[n]{|1 + \frac{1}{z^n}|} = 1$ and thus $\lim_{n\to\infty} \sqrt[n]{|1 + z^n|} = |z|$. So,

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{|1 + z^n|}} = \frac{1}{|z|} < 1$$

meaning that the series $\sum \frac{1}{1+z^n}$ converges. Hence, the series converges iff $|z| > 1$.

Rudin 7. We claim that $\frac{\sqrt{a_n}}{n} < a_n + \frac{1}{n^2}$. Observe that $(n\sqrt{a_n} - 1)^2 > 0$, so $n^2 a_n - 2n\sqrt{a_n} + 1 > 0$ and so $\frac{2\sqrt{a_n}}{n} < a_n + \frac{1}{n^2}$. Thus, by the Comparison Test (as both sides are real and positive), $\sum \frac{2\sqrt{a_n}}{n}$ converges, and so $\sum \frac{\sqrt{a_n}}{n}$ converges.

Rudin 9.a. Note that $\lim_{n\to\infty} \sqrt[n]{n^3} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^3 = 1^3 = 1$. So, the radius of convergence is 1.

Rudin 9.b. Observe $\lim_{n\to\infty} \frac{2}{\sqrt[n]{n!}} = 0$ as the Ratio Test gives

$$\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2}{n+1} = 0.$$

So the radius of convergence is ∞ and the sequence always converges.

9.c. We have:

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}} = \lim_{n\to\infty} \frac{2}{\left(\sqrt[n]{n}\right)^2} = \frac{2}{1^2} = 2.$$

So, the radius of convergence is $\frac{1}{2}$. Note the series is absolutely convergent when $|z| = \frac{1}{2}$ as $\left|\frac{2^n z^n}{n^2}\right| = \frac{1}{n^2}$ whose series converges.

9.d. We have:

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{\left(\sqrt[n]{n}\right)^3}{3} = \frac{1^3}{3} = \frac{1}{3}.$$

So the radius of convergence is 3.

11.a. We wish to show that $\sum a_n$ diverges iff $\sum \frac{a_n}{1+a_n}$ diverges. We will show that $\sum a_n$ converges iff $\sum \frac{a_n}{1+a_n}$ converges. Note as $a_n^2 \geq 0$, $a_n \leq a_n(1 + a_n)$, so $\frac{a_n}{1+a_n} \leq a_n$, implying that if $\sum a_n$ converges, then so does $\sum \frac{a_n}{1+a_n}$. We will now prove the other direction.

Assume that $\sum \frac{a_n}{1+a_n}$ converges. Then $\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0$. Since that $\frac{a_n}{1+a_n} = 1 - \frac{1}{1+a_n}$, $\lim_{n\to\infty} \frac{1}{1+a_n} = 1$. Hence, $\lim_{n\to\infty}(1 + a_n) = 1$ and so $\lim a_n = 0$. This means a_n is bounded; let M be that bound, so $|a_n| \leq M$ for all n. Then for all n,

$$\begin{aligned} a_n &\leq M \\ a_n + 1 &\leq M + 1 \\ 1 &\leq \frac{M+1}{a_n + 1} \\ a_n &\leq \frac{(M+1)a_n}{a_n + 1} \end{aligned}$$

as $a_n > 0$. Note that $\sum \frac{(M+1)a_n}{1+a_n}$ converges since $\sum \frac{a_n}{1+a_n}$ converges. Thus, because of this inequality and as $a_n > 0$, the comparison test implies that $\sum a_n$ converges.

11.b. Note that for all $N \leq i \leq k$, $s_i \leq s_{N+k}$ as (a_n) is positive and so (s_n) is increasing. Therefore, $\frac{1}{s_{N+k}} \leq \frac{1}{s_i}$ and so

$$\sum_{i=N}^{N+k} \frac{a_i}{s_i} \geq \sum_{i=N}^{N+k} \frac{a_i}{s_{N+k}} = \frac{1}{s_{N+k}} \sum_{i=N}^{N+k} a_i = \frac{1}{s_{N+k}} \left(\sum_{i=1}^{N+k} a_i - \sum_{i=1}^{N} a_i \right) = \frac{1}{s_{N+k}} (s_{N+k} - s_N) = 1 - \frac{s_N}{s_{N+k}}.$$

Notice also that $\lim_{k\to\infty} \frac{s_N}{s_{N+k}} = 0$ as $\lim_{k\to\infty} s_{N+k} = \infty$ as $\sum a_n$ diverges. It follows that $\sum \frac{a_n}{s_n}$ does not satisfy the Cauchy criterion.

Let $\epsilon > 0$ such that $0 < \epsilon < 1$. For any N, note that $\lim_{k\to\infty} \frac{s_N}{s_{N+k}} = 0$, meaning there exists some K such that whenever $k \geq K$,

$$\left| \frac{s_N}{s_{N+k}} \right| < 1 - \epsilon$$

Then $1 - \frac{s_N}{s_{N+k}} > \epsilon$. Thus, there exists a $k \geq K$ such that

$$\sum_{i=N}^{N+k} \frac{a_i}{s_i} > 1 - \frac{s_N}{s_{N+k}} > \epsilon$$

so the sum cannot satisfy the Cauchy criterion, as there exists an $\epsilon > 0$ such that for any N there will be a point (actually, infinitely many points) where the Cauchy criterion is not met. Therefore, $\sum \frac{a_n}{s_n}$ diverges.

11.c. We have, for $n > 1$

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_{n-1}s_n} = \frac{a_n}{s_{n-1}s_n}.$$

As $s_{n-1} \leq s_n$ as the sequence is increasing, $\frac{1}{s_{n-1}} \geq \frac{1}{s_n}$ and so

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}.$$

Thus, we have:

$$\begin{aligned}
\sum_{i=1}^{n} \frac{a_i}{s_i^2} &= a_1 + \sum_{i=1}^{n} \frac{a_i}{s_i^2} \\
&\leq a_1 + \sum_{i=2}^{n} \left(\frac{1}{s_{i-1}} - \frac{1}{s_i} \right) \\
&= a_1 + \sum_{i=2}^{n} \frac{1}{s_{i-1}} - \sum_{i=2}^{n} \frac{1}{s_i} \\
&= a_1 + \sum_{i=1}^{n-1} \frac{1}{s_i} - \sum_{i=2}^{n} \frac{1}{s_i} \\
&= a_1 + \frac{1}{s_1} - \frac{1}{s_n}.
\end{aligned}$$

Observe that $\lim_{n\to\infty} \frac{1}{s_n} = 0$ as $\lim_{n\to\infty} s_n = +\infty$. Thus:

$$\sum_{i=1}^{\infty} \frac{a_i}{s_i^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{a_i}{s_i^2} \leq a_1 + \frac{1}{s_1} - \lim_{n\to\infty} \frac{1}{s_n} = a_1 + \frac{1}{s_1}$$

implying that the series $\sum \frac{a_i}{s_i^2}$ converges. Also note that $s_1 = a_1$, so this also shows $a_1 + \frac{1}{a_1} = \frac{1+a_1}{a_1}$ is an upper bound for this series.

11.d. The first series, $\sum \frac{a_n}{1+na_n}$ does not necessarily converge or diverge. For an example where the series diverges, consider $a_n = 1$; this series then becomes $\sum \frac{1}{n+1}$ which diverges.

For an example where the sum diverges, consider:

$$a_n = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{else.} \end{cases}$$

For any $k^2 \leq n < (k+1)^2$,

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{k} 1 = k = \lfloor \sqrt{n} \rfloor.$$

It can be shown that $\lfloor \sqrt{n} \rfloor$ diverges to ∞, so $\sum a_i$ diverges.

Now, if we consider $\sum \frac{a_n}{1+na_n}$ observe that the term $\frac{a_n}{1+na_n}$ is nonzero iff $a_n \neq 1$, which occurs iff n is a square. So,

$$\sum_{i=1}^{n} \frac{a_i}{1+na_i} = \sum_{i=1}^{k} \frac{1}{i^2+1} \leq \sum_{i=1}^{k} \frac{1}{i^2} \leq \sum_{i=1}^{\infty} \frac{1}{i^2}.$$

Hence, $\sum \frac{a_i}{1+na_i}$ converges.

For the second series, note that $\frac{a_n}{1+n^2 a_n} \leq \frac{a_n}{n^2 a_n} = \frac{1}{n^2}$, so by the comparison test, this series diverges.