104 Set 5

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Ross 13.3.a. We show d is a metric on B. Observe for any $x, y \in B$, $d(x, y) \geq 0$ as 0 is a lower bound for the set $\{|x_i - y_i| \mid i > 0\}$.

For the first axiom, take any $x, y \in B$. If $x = y$ then $x_j = y_j$ and so $|x_j - y_j| = 0$; hence $d(x, y) = \sup\{0\} = 0$. On the other hand, if $d(x, y) = 0$, then for all $i > 0$, $|x_i - y_i| \leq 0$, meaning that $x_i = y_i$ and hence $x = y$.

For symmetry, note for any $x, y \in B$, we have:

$$d(x, y) = \sup\{|x_i - y_i| \mid i > 0\} = \sup\{|y_i - x_i| \mid i > 0\} = d(y, x)$$

as in general, $|a - b| = |b - a|$.

For the triangle inequality, take any $x, y, z \in B$. Note that for any $i > 0$, the triangle inequality for the metric on $\mathbb{R}$ gives $|x_i - y_i| + |y_i - z_i| \geq |x_i - z_i|$ and hence $d(x, y) + d(y, z) \geq |x_i - z_i|$. Therefore, $d(x, y) + d(y, z)$ is an upper bound for the set $\{|x_i - z_i| \mid i > 0\}$ and thus $d(x, y) + d(y, z) \geq d(x, z)$.

Ross 13.3.b. Unlike the previous metric, this function is not necessarily real-valued: the distance between the bounded sequences (1) and (0) is ∞. Assuming we allow metrics to give infinite distances as in this example, then yes, this function is a metric.

Ross 13.5.a. Let $\mathcal{U}$ be a nonempty family of sets. Define:

$$A = \bigcap\{S \setminus U \mid U \in \mathcal{U}\}$$

and

$$B = S \setminus \bigcup \mathcal{U}.$$

We will show $A = B$. First, take any $x \in A$. As $\mathcal{U}$ is nonempty, there exists some $U \in \mathcal{U}$; hence, $x \in S \setminus U$ and so $x \in S$. Now, as $x \in A$, for all $U \in \mathcal{U}$, $x \in S \setminus U$ and thus, as $x \in S$, $x \notin U$. As for all $U \in \mathcal{U}$, $x \notin \mathcal{U}$, there does not exist a $U \in \mathcal{U}$ such that $x \in \mathcal{U}$. Therefore, $x \notin \bigcup \mathcal{U}$. Thus, we have $x \in B$.

On the other hand, take the case that $x \in B$. Then $x \in S$ and $x \notin \bigcap \mathcal{U}$, so there does not exist a $U \in \mathcal{U}$ with $x \in U$. Hence, for all $U \in \mathcal{U}$, $x \notin U$. As $x \in S$, $x \in S \setminus U$ for all $U \in \mathcal{U}$. Therefore, $x \in A$ as $\{S \setminus U \mid U \in \mathcal{U}\}$ is nonempty.

Ross 13.5.b. Let $\mathcal{U}$ be a family of closed sets in a topological space (X, T). We will show that the intersection of $\mathcal{U}$ is itself closed. First, observe if $\mathcal{U}$ is empty, its intersection is empty, which is closed as X is an open set and $X \setminus X =$ is thus closed.

Now we take the case that $\mathcal{U}$ is nonempty. For any $U \in \mathcal{U}$, we have that $X \setminus U$ is open, by definition of a closed set. Hence

$$\bigcup\{X \setminus U \mid U \in \mathcal{U}\}$$

is open, being a union of open sets. Thus,

$$X \setminus \bigcup\{X \setminus U \mid U \in \mathcal{U}\} = \bigcap\{X \setminus (X \setminus U) \mid U \in \mathcal{U}\} = \bigcap\{U \mid U \in \mathcal{U}\} = \bigcap \mathcal{U}$$

is closed, applying the result from the previous problem.

Ross 13.3.7. We first prove a lemma.

Lemma. *Let (a, b) and (c, d) be two open intervals with a nonempty intersection. Then their union is $(\min(a, c), \max(b, d))$.*

Proof. For any $x \in (a,b) \cup (c,d)$ observe that $a \leq x \leq b$ or $c \leq x \leq d$ and so $x \geq \min(a,c)$ and $x \leq \max(b,d)$; thus $x \in (\min(a,c), \max(b,d))$. Now we show the other direction.

Take any $x \in (\min(a,c), \max(b,d))$. Let z be a common element in (a,b) and (c,d). If $x \leq z$, then if $\min(a,c) = a$, we have $a < x \leq z < b$ so $a \in (a,b)$. if $\min(a,c) = c$ then $c < x \leq z < d$ so $x \in (c,d)$. Similarly, now take the case that $x > z$. Then if $\max(b,d) = b$ we have $a < z < x < b$ and so $x \in (a,b)$; if $\max(b,d) = d$ we have $c < z < x < d$ and so $x \in (c,d)$. Hence, for any $x \in (\min(a,c), \max(b,d))$ we have $x \in (a,b) \cup (b,d)$. Thus, $(\min(a,c), \max(b,d)) = (a,b) \cup (b,d)$. $\square$

Let E be any open set in $\mathbb{R}$. By definition, for any $x \in E$, there exists some open interval in E containing x. Define

$$A_x = \inf\{a \mid \text{there exists some open interval } (a,b) \text{ with } x \in (a,b) \subseteq E\}$$

and

$$B_x = \sup\{b \mid \text{there exists some open interval } (a,b) \text{ with } x \in (a,b) \subseteq E\}.$$

We allow inf and sup to take on the values $\pm\infty$. Both of these are defined since both sets are defined as there exists an open interval in E which contains x.

Now we claim that $x \in (A_x, B_x) \subseteq E$. For the first part of the statement, notice that as there exists some open interval (a,b) containing x in E, by definition of A_x and B_x, we have $A_x < a < x < b < B_x$ and thus $x \in (A_x, B_x)$.

Now for the second part. Take any $y \in (A_x, B_x)$; we will show $y \in E$. As $A_x < y$, there must exist some $a \geq A_x$ such that $a < y$ and (a,b) is an open interval with $x \in (a,b) \subseteq E$ for some b. This must be true as otherwise, y would be a lower bound for the set defining A_x; but this is impossible as $y > A_x$. Similarly, there exists some $d < B_x$ such that $d > y$ and (c,d) is an interval in E containing x. Observe that $a < y < d$, so $y \in (\min(a,c), \max(b,d)) = (a,b) \cup (c,d) \subseteq E$. Hence, $y \in E$ and so $(A_x, B_x) \subseteq E$.

We define A_x and B_x in this way for all $x \in E$. Now we claim that any two elements in the family of sets $((A_x, B_x))_{x \in E}$ are either disjoint or equal. For, take any $x, y \in E$ and assume that (A_x, B_x) and (A_y, B_y) are not disjoint. Then we must have $(\min(A_x, A_y), \max(B_x, B_y)) = (A_x, B_x) \cup (A_y, B_y)$ by our lemma, so $x, y \in (\min(A_x, A_y), \max(B_x, B_y)) \subseteq E$. However, by minimality of A_x we have $A_x \leq \min(A_x, A_y) \leq A_x$ and thus $\min(A_x, A_y) = A_x$. Similarly, $A_y = \min(A_x, A_y) = A_x$ and $B_x = \min(B_x, B_y) = B_y$. Hence, $(A_x, B_x) = (A_y, B_y)$.

Observe now that

$$E = \bigcup \{(A_x, B_x) \mid x \in E\}$$

as for each $x \in E$, $(A_x, B_x) \subseteq E$ so the union is a subset of E; also, for any $x \in E$, $x \in (A_x, B_x)$ and so x is in this union. Note that any two distinct elements in this set are disjoint as they are not equal and that each element is an open interval. Hence, we have proven our claim.

4. We have already shown that $S \subseteq \bar{S}$; thus $\bar{S} \subseteq \bar{\bar{S}}$. Now we show the reverse direction.

For any $p \in \bar{\bar{S}}$ there exists some sequence (x_n) in $\bar{S}$ such that $\lim x_n = p$. For each $x_n \in \bar{S}$, there exist some sequence $(y_k)_k$ such that $\lim y_k = x_n$. Thus, there exists some $N > 0$ such that if $k \geq N$, then $d(y_k, x_n) < \frac{1}{n}$. Let $z_n = y_k$ for some $k > K$; construct a sequence (z_n) of elements of S in this way. Note that $d(z_n, x_n) < \frac{1}{n}$ for all $n > 0$.

We now claim that $\lim z_n = p$. To see this, take any $\epsilon > 0$. Since $\lim x_n = p$, there exists some $N > 0$ such that if $n \geq N$, $d(x_n, p) < \frac{\epsilon}{2}$. Since $\lim \frac{1}{n} = 0$, there exists some $M > 0$ such that if $n \geq M$, $\frac{1}{n} < \frac{\epsilon}{2}$ so $d(x_n, z_n) < \frac{\epsilon}{2}$. Hence, if $n \geq \max(N, M)$ then $d(z_n, p) \leq d(x_n, z_n) + d(x_n, p) = d(z_n, x_n) + d(x_n, p) < \epsilon$. Therefore, $\lim z_n = p$. As (z_n) is a sequence in S, this means $p \in \bar{S}$, so $\bar{S} \subseteq S$ and thus $S = \bar{S}$.

5. Let T be the intersection of all closed sets containing S. Note that we are taking an intersection of elements of a nonempty set as $\bar{S}$ is one such set.

Now, as we have shown earlier in this homework, the intersection of closed sets is itself closed, and thus T is closed. Observe also, as $\bar{S}$ is a closed set containing S, $T \subseteq \bar{S}$. Now, as T is closed, $\bar{T} = T$. As $S \subseteq T$, any limits of sequences in S will be in $\bar{T}$. Thus, $\bar{S} \subseteq \bar{T}$, and so $\bar{T} = \bar{S}$. Thus, $\bar{S}$ is the smallest closed set containing S (or the closed set generated by S).