104 Set 6

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Problem 1. $[0,1]^2 \subseteq \mathbb{R}^2$ *is sequentially compact.*

First, we show a lemma.

Lemma 1. *A sequence in* $\mathbb{R}^2$ *converges if and only if it converges in* $\mathbb{R}$.

Proof. $\Longrightarrow$ Let $((a_n, b_n))_n$ be a sequence in $\mathbb{R}^2$ that converges to (a, b). We will show that (a_n) converges to a and (b_n) converges to b. Take any $\epsilon > 0$. Then there exists an $N > 0$ such that if $n \geq N$

$$\epsilon > |(a_n, b_n) - (a, b)| = \sqrt{(a_n - a)^2 + (b_n - b)^2} \geq |a_n - a|, |b_n - b|.$$

It follows that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. $\Longleftarrow$ Let (a_n) and (b_n) be sequences in $\mathbb{R}$ that converge to a and b respectively. We will show that $((a_n, b_n))$ converges to (a, b). For any $\epsilon > 0$, there exists an $N > 0$ such that if $n \geq N$, then $|a_n - a| < \frac{\epsilon}{2}$ and an $M > 0$ such that if $n \geq M$, $|b_n - b| < \frac{\epsilon}{2}$. Hence, we have $|a_n - a| + |b_n - b| < \epsilon$ whenever $n \geq \max(N, M)$.

Observe now that:

$$\begin{aligned}
|a_n - a||b_n - b| &\geq 0 \\
2|a_n - a||b_n - b| &\geq 0 \\
(a_n - a)^2 + (b_n - b)^2 + 2|a_n - a||b_n - b| &\geq (a_n - a)^2 + (b_n - b)^2 \\
(|a_n - a| + |b_n - b|)^2 &\geq (a_n - a)^2 + (b_n - b)^2 \\
|a_n - a| + |b_n - b| &\geq \sqrt{(a_n - a)^2 + (b_n - b)^2}
\end{aligned}$$

as both sides are nonnegative. Hence, it follows that $|(a_n, b_n) - (a, b)| < \epsilon$, and so $\lim_{n\to\infty}(a_n, b_n) = (a, b)$. □

Problem 1 follows.

Proof of Problem 1. Take any sequence $((a_n, b_n))$ in $[0,1]^2$. Then (a_n) is a sequence in $[0,1]$, so by Bolzano-Weierstrass, there exists a subsequence $(a_{r_n})_n$ that converges to a. Note $a \in [0,1]$. Note now that (b_{r_n}) is a sequence in $[0,1]$, so there exists a subsequence $(b_{r_{s_n}})_n$ that converges to $b \in [0,1]$. Note also that $(a_{r_{s_n}})$ converges to a as it is a subsequence of (a_{r_n}). Thus, $((a_{r_{s_n}}, b_{r_{s_n}}))$ converges to $(a, b) \in [0,1]^2$ and so $((a_n, b_n))$ has a subsequence that converges in $[0,1]^2$. Thus, $[0,1]^2$ is compact in $\mathbb{R}^2$. □

Problem 2. E *is uncountable and compact.*

Proof. To see that E is uncountable, consider an injection $[0,1] \to E$ such that for any $x \in [0,1]$, x maps to the decimal number formed from the binary representation of x, where every 0 is replaced with 4 and 1 replaced with 7. Note this is an element of E, as it contains only 4's and 7's. This map is also injective since if any two sequences differ in at least one place, then they will be different numbers. It follows that E is uncountable, for if E is countable, then this would imply $[0,1]$ is either finite or countably infinite, which is impossible.

To show E is compact, we show it is both closed and bounded. By Heine-Borel, this will imply that E is compact (in particular, being a subset of a closed interval, which is compact). It is clear that E is bounded, being a subset of $[0,1]$, by definition.

Showing closure is trickier. Take any sequence (a_n) in E that converges to some $a \in \mathbb{R}$. Note as $E \subseteq [0,1]$, $a \in [0,1]$. Let the decimal expansion of a_n be:

$$a_n = \sum_{k=1}^{\infty} \frac{d_{nk}}{10^k}$$

and the decimal expansion of a be:

$$a = \sum_{k=1}^{\infty} \frac{d_k}{10^k}.$$

We will show that for each $r > 0$, there exists some $N > 0$ such that for all $n \geq N$, $d_{nr} = d_r$. To do this, observe by the limit definition that there exists some $N > 0$ such that $|a_n - a| < \frac{1}{10^r}$. Then we have:

$$\begin{aligned}
\frac{1}{10^r} &> |a_n - a| \\
&= \left| \sum_{k=1}^{\infty} \frac{d_{nk}}{10^k} - \sum_{k=1}^{\infty} \frac{d_k}{10^k} \right| \\
&= \left| \sum_{k=1}^{\infty} \frac{d_{nk} - d_k}{10^k} \right| \\
&\geq \left| \sum_{k=1}^{r} \frac{d_{nk} - d_k}{10^k} \right| + \left| \sum_{k=r+1}^{\infty} \frac{d_{nk} - d_k}{10^k} \right| \\
&\geq \left| \sum_{k=1}^{r} \frac{d_{nk} - d_k}{10^k} \right| \\
&\geq \sum_{k=1}^{r} \frac{|d_{nk} - d_k|}{10^k}.
\end{aligned}$$

So, for each $1 \leq k \leq r$, we have $\frac{|d_{nk} - d_k|}{10^k} < \frac{1}{10^r}$ and so $|d_{nk} - d_k| < 10^{k-r}$. Note as $k \leq r$, $k - r \leq 0$, so $|d_{nk} - d_k| < 10^{k-r} < 1$. As $|d_{nk} - d_k| \geq 0$ is an integer, this implies $|d_{nk} - d_k| = 0$, and so $d_k = d_{nk}$. It thus follows that the decimal expansion of a consists only of 4 and 7, as each digit of a is a digit of infinitely many a_n.

Thus, $a \in E$, and so E is closed. Heine-Borel implies that E is compact. $\square$

Problem 3. *There exists a sequence of sets $A_1, A_2, \ldots \subseteq X$ where X is a metric space, such that $\overline{\bigcup_{k=1}^{\infty} A_k} \neq \bigcup_{k=1}^{\infty} \overline{A_k}$.*

Proof. Consider the metric space $\mathbb{R}$ and the sets $A_1, A_2, \ldots$ such that:

$$A_k = \left\{ \frac{1}{k+1} + a \;\middle|\; a \in \mathbb{Z} \right\}.$$

Observe that the minimum distance between two different points in A_k is 1. This implies that if a sequence (a_n) in A_k converges to a, then by the Cauchy criternion there exists an $N > 0$ such that whenever $m, n \geq N$, $|a_n - a_m| < \frac{1}{2}$, implying that $a_n = a_m$ or else the distance between a_n and a_m is at least 1. (Another way to see this is to note that every singleton set in A_k is open and thus the induced topology on A_k is discrete.)

So, any sequence in A_k eventually becomes constant. Thus, it will converge to this constant value, which will be in A_k. This implies A_k is closed, so $\overline{A_k} = A_k$.

However, observe that $\bigcup_{k=1}^{\infty} A_k$ contains the sequence $(\frac{1}{n})_{n=2}^{\infty}$, which converges to 0. Thus, 0 is in the closure of this set. However, 0 is not in $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} \overline{A_k}$, since 0 is not contained in any of the A_k's. Thus,

$$\overline{\bigcup_{k=1}^{\infty} A_k} \neq \bigcup_{k=1}^{\infty} \overline{A_k}$$

$\square$

4. To see this this argument does not lead to the conclusion, let us continue this argument. Assume that $E \subseteq \mathbb{R}$ is our closed set. Then we can write

$$X \setminus E = \bigcup_{k=1}^{\infty} E_k$$

for some disjoint open intervals $E_1, E_2, \ldots$, some of them possibly empty. Then we have:

$$E = X \setminus \bigcup_{k=1}^{\infty} E_k = \bigcap_{k=1}^{\infty} X \setminus E_k.$$

It should be clear at this point that this argument will not show that E is the union of closed intervals, since this results in an intersection. We might also note that the sets $X \setminus E_k$ will be in the form $(-\infty, a] \cup [b, +\infty)$ if $E_k = (a, b)$. Also, for any E_k and $E_{k'}$, assuming these are bounded open intervals, their complements will intersect. So this is not satisfactory in proving the claim.

Now, for a counterexample, consider the set E from problem 2 above. We claim that no closed interval $[a, b] \subseteq E$, where $a < b$. Otherwise, we might find some k such that $\frac{1}{10^k} \leq b - a$ so $a \leq a + \frac{1}{10^k} \leq b$. But this means $a, a + \frac{1}{k} \in E$, which is a contradiction, since the digit in the kth position of a will be either 0, 4, or 7, meaning that the kth digit in $a + \frac{1}{k}$ will be either 1, 5, or 8, none of which are allowed in elements of E.

So, if E is written as a union of closed intervals, every interval must be a singleton set. For a countable union, this would imply there exists a bijection from a countable set to E, which is impossible as E is uncountable.