104 Set 7

Ishaan Patkar

1. Let X and Y be compact topological spaces. Then $X \times Y$ is compact, under the product topology.

Proof. Take any open cover $\{E_\alpha\}_{\alpha \in A}$ of $X \times Y$. For every $(x, y) \in X \times Y$, there exists some $\alpha \in A$ such that $(x, y) \in E_\alpha$. As E_α is open, by definition of the product topology, there exists some $U_{xy} \times V_{xy} \subseteq E_\alpha$ with $U_{xy} \subseteq X$, $V_{xy} \subseteq Y$, open, and $(x, y) \in U_{xy} \times V_{xy}$. Notice that $\{U_{xy} \times V_{xy}\}_{(x,y) \in X \times Y}$ is an open cover of $X \times Y$.

Now, take any $y \in Y$. Then observe that $\{U_{xy}\}_{x \in X}$ covers the set X: for any $x \in X$, as $(x, y) \in U_{xy} \times V_{xy}$, $x \in U_{zy}$. Since X is compact, there exists a finite subcover $U_{r_1}, U_{r_2}, \ldots, U_{r_n}$, where $r_i = (x_i, y)$ for some $x_i \in X$. Now let $W_y = \bigcap_{i=1}^n V_{r_i}$. Observe that W_y is open in Y as this is a finite intersection of open sets in Y; it is also nonempty since $y \in V_{r_i}$ for each i, and thus $y \in W_y$.

As $y \in W_y$ for all $y \in Y$, $\{W_y\}_{y \in Y}$ is an open cover of Y. As Y is compact, there is a finite subcover $W_{y_1}, W_{y_2}, \ldots, W_{y_m}$. For each y_j, there exists an open cover of X, $U_{x_{1j}, y_j}, U_{x_{2j}, y_j}, \ldots, U_{x_{n_j, j}, y_j}$, as defined above. Observe that $W_{y_j} = \bigcap_{i=1}^{n_j} V_{x_{ij}, y_j}$. Thus:

$$X \times W_{y_j} \subseteq \bigcup_{i=1}^{n_j} U_{x_{ij}, y_j} \times W_y \subseteq \bigcup_{i=1}^{n_j} U_{x_{ij}, y_j} \times V_{x_{ij}, y_j}$$

as $W_y \subseteq V_{x_{ij}, y_j}$ for each i. As $W_{y_1}, W_{y_2}, \ldots, W_{y_m}$ covers Y, we then have:

$$X \times Y \subseteq \bigcup_{j=1}^{m} X \times W_y \subseteq \bigcup_{j=1}^{m} \bigcup_{i=1}^{n_j} U_{x_{ij}, y_j} \times V_{x_{ij}, y_j}.$$

Note this is a finite subcovering. By definition of the U's and V's, for each (x_{ij}, y_j), there exists some $\alpha_{ij} \in A$ such that $U_{x_{ij}, y_j} \times V_{x_{ij}, y_j} \subseteq E_{\alpha_{ij}}$. Therefore, we have:

$$X \times Y \subseteq \bigcup_{j=1}^{m} \bigcup_{i=1}^{n_j} E_{\alpha_{ij}}$$

which is a finite subcovering of $\{E_\alpha\}_{\alpha \in A}$. Hence, $X \times Y$ is compact. □

2.a. *If A is open, then $f(A)$ is open.* False. Consider the constant map $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for all $x \in \mathbb{R}$. Then for any open set $E \subseteq \mathbb{R}$, $f(E) = \{0\}$ which is not open, since no open ball centered at 0 is contained within $f(E)$.

2.b. *If A is closed, then $f(A)$ is closed.* False. Let $f : \mathbb{R} \to \mathbb{R}$ be the map $f(x) = x^2$. Observe that $\mathbb{R}$ is closed, but $f(\mathbb{R}) = [0, +\infty)$ is not, since no open ball containing 0 is contained within $[0, +\infty)$.

2.c. *If A is bounded, then $f(A)$ is bounded.* False. Let $f : (0, 1) \to \mathbb{R}$ be the continuous map $f(x) = \frac{1}{x}$. Then $f((0, 1)) = (1, +\infty)$ which is not bounded, even though $(0, 1)$ is bounded.

2.d. *If A is compact, then $f(A)$ is compact.* True. Take any open cover $\{E_\alpha\}_{\alpha \in I}$ of $f(A)$. Observe that $\{f^{-1}(E_\alpha)\}_{\alpha \in I}$ is an open cover of A, since if $x \in A$, $f(x) \in f(A)$, so there exists an $i \in I$ with $f(x) \in E_i$, so $x \in f^{-1}(E_i)$. Then there exists a finite subcover $f^{-1}(E_{r_1}), f^{-1}(E_{r_2}), \ldots, f^{-1}(E_{r_n})$ of A. Then we claim $E_{r_1}, E_{r_2}, \ldots, E_{r_n}$ is an open cover of $f(A)$, since, for any $y \in f(A)$, there exists an $x \in A$ with $f(x) = y$. So, $x \in f^{-1}(E_{r_i})$ for some i, and so $y = f(x) \in E_{r_i}$. Thus, $f(A)$ is contained in the union of the E_{r_i}'s.

2.e. *If A is connected, then $f(A)$ is connected.* True. Let $f : X \to Y$ be continuous, and $A \subseteq X$ connected. Suppose that $f(A)$ is disconnected. Then there exist open sets $U, V \subseteq Y$ such that $f(A) \cap U$ and $f(A) \cap V$ are nonempty, disjoint, and $f(A) \subseteq (f(A) \cap U) \cup (f(A) \cap V)$.

We claim now that $f(A) \cap f^{-1}(U)$ and $f(A) \cap f^{-1}(V)$ are disjoint, nonempty, and open in A, and whose union contains A. First, note they are disjoint since if $x \in A \cap f^{-1}(U) \cap f^{-1}(V)$, then $f(x) \in f(A), U, V$, and so $f(x) \in (f(A) \cap U) \cap (f(A) \cap V)$, which is impossible, as $f(A) \cap U$ and $f(A) \cap V$ are disjoint. Since $f(A) \cap U$ is nonempty, there exists some $y \in f(A) \cap U$. As $y \in f(A)$, there exists an $x \in A$ with $f(x) = y$, so as $y \in U$, $x \in f^{-1}(U)$. So, $x \in A \cap f^{-1}(U)$. Similarly, $A \cap f^{-1}(V)$ is nonempty. Now, $f^{-1}(U)$ and $f^{-1}(V)$ are open, since U and V are open in Y and f is continuous, so $A \cap f^{-1}(U)$ and $A \cap f^{-1}(V)$ are open in A. Finally, $A \subseteq (A \cap f^{-1}(U)) \cup (A \cap f^{-1}(V))$ since for any $x \in A$, $f(x) \in f(A) \subseteq (f(A) \cap U) \cup (f(A) \cap V)$, so either $f(x) \in U$ or $f(x) \in V$, implying that $x \in f^{-1}(U)$ or $x \in f^{-1}(V)$.

But this means A is disconnected, which is impossible. So, $f(A)$ must be connected (since $f(A) \neq \emptyset$ since $A \neq \emptyset$).

3. There does not exist a continuous surjection $f : [0, 1] \to \mathbb{R}$.

Proof. This follows from 2.*d*: observing that $[0, 1]$ is compact, if there existed such a map, then $f([0, 1]) = \mathbb{R}$ would be compact, which it is not. (For instance, the open covering $\{(n - 1, n + 1)\}_{n \in \mathbb{Z}}$ has no finite subcovering, as any subcovering must contain every integer, and every integer is contained in a subcovering, but every integer n is contained only in the interval $(n - 1, n + 1)$, and so this interval must belong in the subcovering. But this leads to an infinite number of subcoverings. Alternatively, one might see this by noting that if $\mathbb{R}$ could be written as a finite subcovering of these open sets, then $\mathbb{R}$ would be bounded, which leads to contradiction.) □