104 Set 8

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1. *Let (f_n) be a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ with $f_n = \frac{n+\sin x}{2n+\cos(n^2x)}$. Then f_n uniformly converges to $\frac{1}{2}$.*

Proof. For any $n \geq 1$, consider $d_\infty(f_n, \frac{1}{2})$. We have, for all $x \in \mathbb{R}$:

$$\begin{aligned} d\left(f_n(x), \frac{1}{2}\right) &= \left|\frac{n+\sin x}{2n+\cos(n^2x)} - \frac{1}{2}\right| \\ &= \left|\frac{2n+\sin x}{4n+2\cos(n^2x)} - \frac{2n+\cos(n^2x)}{4n+2\cos(n^2x)}\right| \\ &= \left|\frac{\sin x - \cos(n^2x)}{4n+2\cos(n^2x)}\right| \\ &= \frac{|\sin x - \cos(n^2x)|}{|4n+2\cos(n^2x)|}. \end{aligned}$$

Note that $4n + 2\cos(n^2x) \geq 4n - 2$. As $n \geq 1$, $4n + 2\cos(n^2x) \geq 4n - 2 \geq 0$, so $|4n + 2\cos(n^2x)| = 4n + 2\cos(n^2x)$. Also, we have $|\sin x - \cos(n^2x)| \leq 3$. Hence:

$$d\left(f_n(x), \frac{1}{2}\right) = \frac{|\sin x - \cos(n^2x)|}{|4n+2\cos(n^2x)|} = \frac{|\sin x - \cos(n^2x)|}{4n+2\cos(n^2x)} \leq \frac{3}{4n-2}.$$

Hence, $d_\infty(f_n, \frac{1}{2}) \leq \frac{3}{4n-2}$. Observe that:

$$\lim_{n\to\infty} \frac{3}{4n-2} = \lim_{n\to\infty} \frac{\frac{3}{n}}{4-\frac{2}{n}} = \frac{\lim_{n\to\infty} \frac{3}{n}}{\lim_{n\to\infty}(4-\frac{2}{n})} = \frac{0}{4-0} = 0.$$

Hence, for any $\epsilon > 0$, there exists an $N > 0$ such that $\left|\frac{3}{4n-2}\right| < \epsilon$ whenever $n \geq N$. As $4n-2 \geq 0$, $\frac{3}{4n-2} \geq 0$, meaning that $\frac{3}{4n-2} < \epsilon$. Hence:

$$d_\infty\left(f_n, \frac{1}{2}\right) \leq \frac{3}{4n-2} < \epsilon$$

whenever $n \geq N$. Therefore, f_n converges uniformly to $\frac{1}{2}$. □

Note we might also have shown uniform convergence using the Weierstrass M-test.

2. *Let $f(x) = \sum_{n=1}^\infty a_n x^n$, where $\sum_{n=1}^\infty |a_n|$ is convergent. Then f is continuous on $[-1, 1]$.*

Proof. Assume that $f : [-1, 1] \to \mathbb{R}$. For all $n \geq 1$, let $f_n : [-1, 1] \to \mathbb{R}$ such that $f_n(x) = a_n x^n$ for all $x \in [-1, 1]$. Then $f(x) = \sum_{n=1}^\infty f_n(x)$ for any $x \in \mathbb{R}$. We claim that $\sum f_n$ convgerges to f uniformly. We will prove this using the Weierstrass M-test.

For any $n \geq 1$, $x \in [-1, 1]$, observe that $|f(x)| = |a_n x^n| = |a_n||x|^n \leq |a_n|$ as $|x| \leq 1$. Hence, by the Weierstrass M-test, the series $\sum f_n$ uniformly converges. Since $\sum f_n$ converges pointwise to f, it also must uniformly converge to f.

As each f_n is continuous, it follows that for all $m \geq 1$, $\sum_{n=1}^m f_n$ is continuous, since the sum of continuous functions is continuous. Therefore, $f = \lim_{m\to\infty} \sum_{n=1}^m f_n$ must be continuous, as the partial sums are continuous and uniformly converge to f. □

Since $\sum n^{-2}$ converges absolutely, this gives that the power series $\sum_{n=1}^{\infty} n^{-2}x^n$ converges when $x \in [-1, 1]$.

3. *Let $f : (-1, 1) \to \mathbb{R}$ such that $f(x) = \sum_{n=0}^{\infty}$. Then f is continuous.*

Proof. Take any $a \in [0, 1)$, and let $f_n : [-a, a] \to \mathbb{R}$ be a sequence of functions, $n \geq 0$, such that $f_n(x) = x^n$ for all $x \in [-a, a]$. We claim that $\sum f_n$ converges uniformly to $f\,|_{[-a,a]}$. To see this, we first show pointwise convergence: for any $x \in [-a, a]$, $f(x) = \sum_{n=1}^{\infty} x^n = \sum_{n=0}^{\infty} f_n(x)$. Now we show that the series converges uniformly, implying that it converges uniformly to f.

We will do this by the Weierstrass M-test. Observe that $|f_n(x)| = |x^n| = |x|^n \leq a^n$ as $x \in [-a, a]$, meaning $|x| \leq a$. Since $a \in [0, 1)$, the series $\sum a^n$ converges to $\frac{1}{1-a}$. Hence, the series $\sum f_n$ converges uniformly. It must then converge uniformly to $f\,|_{[-a,a]}$, as it converges pointwise to this function.

Now, since each of the f_n are continuous, the partial sums of the series $\sum f_n$ are continuous, meaning that as the partial sums converge uniformly to $f\,|_{[-a,a]}$, it must be continuous. Thus, for any $x \in (-1, 1)$, we can pick some $a \in [0, 1]$ such that $x \in (-a, a)$ (for instance, $a \in (|x|, 1)$). So, $f\,|_{[-a,a]}$ is continuous at x and thus f is continuous at x. Hence, f is continuous. □

Note we might have shown this in a much simpler manner by noting that $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, of which we can check continuity through limit properties. We will use this observation to show that $\sum x_n$ does not converge uniformly to f for $x \in (-1, 1)$. First, we show a lemma.

Lemma 1. *Let (f_n) be a sequence of functions $f : X \to \mathbb{R}$ that converges uniformly to f. If f_n is bounded for every n, then f is bounded.*

Proof. Take any $\epsilon > 0$. Then there exists some N such that whenever $n \geq N$, $d_\infty(f_n, f) < \epsilon$. Note $d_\infty(0, f_n)$ is defined as f_n is bounded. So, $d_\infty(0, f_n) + d_\infty(f_n, f) < \epsilon + d_\infty(0, f_n)$. Hence, by the Triangle Inequality, $d_\infty(0, f) < \epsilon + d_\infty(0, f_n)$. Thus, as $\epsilon + d_\infty(0, f_n) < \infty$, f is bounded since for all $x \in X$, $|f(x)| \leq d_\infty(0, f_n) < \epsilon + d_\infty(0, f_n)$. □

Observe now that x^n is bounded for every n and $x \in (-1, 1)$. Hence, the partial sums $\sum_{n=0}^{N} x^n$ must be bounded: in particular, $\left|\sum_{n=0}^{N} x^n\right| \leq \sum_{n=0}^{N} |x^n| = \sum_{n=0}^{N} |x|^n \leq N + 1$. However, f is unbounded: for any integer $n > 0$, note $f(1 - \frac{1}{n}) = n$. Thus, if $\sum x^n$ converges uniformly to f, it would imply f is bounded, which is impossible. So, the convergence is not uniform.