104 Set 8

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1. There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ when $x \le 0$ and $f(x) = 1$ when $x \ge 1$.

Proof. We use the method outlined in Ross Exercise 31.4. Let

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

be the smooth function defined in the example. Note that $f(1-x)$ is also infinitely differentiable, as $\frac{d^n}{dx^n} f(1-x) = (-1)^n f^{(n)}(1-x)$, and observe that $f(1-x) = 0$ when $x \ge 1$ and $f(1-x) = e^{-\frac{1}{x}}$ when $x < 1$. Hence, $f(x) + f(1-x) > 0$ for all x; when $x < 1$, $f(1-x) > 0$ and $f(x) \ge 0$, and when $x \ge 1$, $f(1-x) = 0$ and $f(x) > 0$. In particular, this function is infinitely differentiable, being a sum of infinitely differentiable functions. Thus, we can define the function:

$$g(x) = \frac{f(x)}{f(x) + f(1-x)}.$$

This is infinitely differentiable since both the numerator and denominator are infinitely differentiable, and the denominator is nonzero everywhere. When $x \le 0$, $f(x) = 0$ so $g(x) = 0$. When $x \ge 1$, $f(1-x) = 0$, so $g(x) = 1$. As g is smooth, we are done. □

Rudin Exercise 5.4. Let $f(x)$ be the function such that:

$$f(x) = \sum_{k=0}^{n} \frac{C_k}{k+1} x^{k+1}.$$

Note we have $f(0) = 0$ as $f(x) = x \sum_{k=0}^{n} \frac{C_k}{k+1} x^k$. We also have $f(1) = \sum_{k=0}^{n} \frac{C_k}{k+1} = 0$ by the statement of the problem. As f is a polynomial, it is differentiable everywhere. Thus, it follows from Rolle's Theorem that there exists some $c \in (0,1)$ such that $f'(c) = 0$. Observe that:

$$f'(x) = \sum_{k=0}^{n} \frac{C_k}{k+1} \frac{d}{dx} x^{k+1} = \sum_{k=0}^{n} C_k x^k$$

and so this means as $f'(c) = 0$, the equation

$$f'(x) = \sum_{k=0}^{n} C_k x^k$$

has a real root c in $(0,1)$.

Rudin Exercise 5.8. Since f' is continuous on $[a,b]$ and the latter is compact, f' is uniformly continuous. Thus, for any $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x, y \in [a,b]$ with $0 < |x-y| < \delta$, then $|f'(x) - f'(y)| < \epsilon$. Hence, for any $x, t \in [a,b]$ with $0 < |x-t| < \delta$. First, take the case that $x < t$. Then as f is differentiable on $[x,t] \subseteq [a,b]$, by the Mean Value Theorem, there exists a $c \in (x,t)$ with

$$f'(c) = \frac{f(x) - f(t)}{x - t}.$$

Thus, as $c \in (x,t)$, $0 < c - x < t - x < \delta$, meaning

$$\left| \frac{f(x) - f(t)}{x - t} - f'(x) \right| = |f'(c) - f'(x)| < \epsilon.$$

The case when $x > t$ follows similarly. Thus, whenever $0 < |x - t| < \delta$,

$$\left| \frac{f(x) - f(t)}{x - t} - f'(x) \right| < \epsilon.$$

Rudin Exercise 5.18. We claim that for any $1 \leq k \leq n-1$:

$$f^{(k)}(t) = kQ^{(k-1)}(t) + (t - \beta)Q^{(k)}(t).$$

For the base case $k = 1$, we have

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

by definition of Q. Then, differentiating and applying the product rule yields

$$f'(t) = Q(t) + (t - \beta)Q'(t).$$

This establishes the base case. Now, take any $1 < k \leq n-1$ and assume this equation holds for $k - 1$. Thus:

$$f^{(k-1)}(t) = (k-1)Q^{(k-2)}(t) + (t - \beta)Q^{(k-1)}(t)$$

and so differentiating gives

$$f^{(k)}(t) = kQ^{(k-1)}(t) + (t - \beta)Q^{(k)}(t).$$

Thus the induction is complete.

Next, we use these equations when $t = \alpha$ to write P, the Taylor approximation, in terms of Q. We have:

$$\begin{aligned}
P(\beta) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k \\
&= f(\alpha) + \sum_{k=1}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k \\
&= f(\alpha) + \sum_{k=1}^{n-1} \frac{kQ^{(k-1)}(\alpha) + (\alpha - \beta)Q^{(k)}(\alpha)}{k!}(\beta - \alpha)^k \\
&= f(\alpha) + \sum_{k=1}^{n-1} \frac{Q^{(k-1)}(\alpha)}{(k-1)!}(\beta - \alpha)^k - \sum_{k=1}^{n-1} \frac{Q^{(k)}(\alpha)}{k!}(\beta - \alpha)^{k+1} \\
&= f(\alpha) + \sum_{k=0}^{n-2} \frac{Q^{(k)}(\alpha)}{k!}(\beta - \alpha)^{k+1} - \sum_{k=1}^{n-1} \frac{Q^{(k)}(\alpha)}{k!}(\beta - \alpha)^{k+1} \\
&= f(\alpha) + \frac{Q(\alpha)}{0!}(\beta - \alpha) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n \\
&= f(\alpha) + (f(\beta) - f(\alpha)) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n \\
&= f(\beta) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n.
\end{aligned}$$

Therefore, we obtain

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n$$

as desired.

Rudin 5.22.

a. Suppose f has two distinct fixed points at $x, y \in \mathbb{R}$. Assume, without loss of generality, that $x < y$. Then $f(x) = x$ and $f(y) = y$. As f is differentiable, by assumption, it is differentiable on $[x, y]$, so by the Mean Value Theorem, there exists a $c \in [x, y]$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x} = \frac{y - x}{y - x} = 1.$$

But this is a contradiction since, by assumption $f'(t) \neq 1$ for all $t \in \mathbb{R}$. Thus, f can have at most 1 fixed point.

b. Suppose that $f(t) = t + (1 + e^t)^{-1}$ has a fixed point at x. Then $f(x) = x$ meaning $x + (1 + e^t)^{-1} = x$ and so $(1 + e^t)^{-1} = 0$, which is impossible as 0 is not in the range of the reciprocal function $t \mapsto t^{-1}$.

c. Pick an $x_0 \in \mathbb{R}$ and define a sequence recursively such that for each $n > 0$, $x_n = f(x_{n-1})$. Let $c = |x_1 - x_0|$. We claim that for each $n \geq 0$, $|x_{n+1} - x_n| \leq cA^n$. We prove this by induction. For the base case, when $n = 0$, observe that $|x_1 - x_0| = c$ by definition. Now, assume this holds for any $n \geq 0$. Consider $n + 1$. Note that $x_{n+2} = f(x_{n+2})$ and $x_{n+1} = f(x_n)$. If $x_n = x_{n+1}$ then $x_{n+2} = f(x_{n+1}) = f(x_n) = x_{n+1}$ and so $0 = |x_{n+1} - x_n| \leq cA^n$.

Otherwise, either $x_n < x_{n+1}$ or $x_n > x_{n+1}$. If $x_n < x_{n+1}$, then as f is differentiable everywhere, including $[x_n, x_{n+1}]$, by the Mean Value Theorem, there exists a $c \in (x_n, x_{n+1})$ with

$$\frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} = f'(c)$$

and so

$$\left| \frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n} \right| = |f'(c)| \leq A.$$

The case with $x_n > x_{n+1}$ leads to the same result. Now we have

$$|x_{n+2} - x_{n+1}| \leq A|x_{n+1} - x_n| \leq A(cA^n) = cA^{n+1}.$$

So the induction is complete.

Now as $0 \leq A < 1$, the series $\sum cA^n$ converges (absolutely). Since $|x_{n+1} - x_n| \leq cA^n$, the series $\sum(x_{n+1} - x_n)$ also converges. In particular, this means the partial sums converge. Note that the partial sums are:

$$\sum_{k=0}^{n-1}(x_{k+1} - x_k) = \sum_{k=1}^{n} x_k - \sum_{k=0}^{n-1} x_k = x_n - x_0.$$

Hence, $\lim_{n\to\infty}(x_n - x_0) = \lim_{n\to\infty} x_n - x_0$ is defined.

Let now $x = \lim_{n\to\infty} x_n$. We claim x is a fixed point. As f is continuous and $x_n \to x$, $f(x_n) \to f(x)$. Thus, $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$ and x is a fixed point.

d. One way to construct this path is to follow the following algorithm. Draw the graph on the plane, and start with an arbitrary x_0 on the real line. Then go up to the point $(x_0, f(x_0))$ on the graph. This point $f(x_0)$ is now x_1. Now go to the point (x_1, x_1) which is the unique point at the height x_1 which forms a square with the origin. Then repeat the process from this point: move vertically to find the point on the function at the same x, then move horizontally to the square, and repeat. Eventually, the squares and the points on the functions will get closer and closer, and the previous part proves they will eventually converge at the limit, which will be a fixed point.