1.10 Prove $(2n+1)+(2n+3)+(2n+5)+\dots+(4n-1)=3n^2$ for all positive integers n.

$4n-1 = (2n-1)+(2n)$, rewrite problem as

$(2n+1)+(2n+3)+(2n+5)+\dots+((2n-1)+(2n)) = 3n^2$

Proof by induction:

base case $n=1$: $3=3$

Inductive Hypothesis: Assume $P(k)$ is true for $n=k$, then

$(2k+1)+(2k+3)+(2k+5)+\dots+(4k-1) = 3k^2$

Inductive step $n=k+1$:

$$\begin{aligned}
LHS &= (2(k+1)+1)+(2(k+1)+3)+(2(k+1)+5)+\dots+(2(k+1)-1+2(k+1)) \\
&= (2k+3)+(2k+5)+(2k+7)+\dots+ \\
&= (2k+1)+2+(2k+3)+2+\dots+(2k+1+2k+2) \\
&= \underbrace{(2k+1)+(2k+3)+\dots+(4k-1)}+2k+(4k+3) \\
&= 3k^2+2k+4k+3 \\
&= 3k^2+6k+3 \\
&= 3(k+1)^2
\end{aligned}$$

hence it's true for $n=k+1$.

1.12 a) $n=1$: $(a+b)^1 = \binom{1}{0}a^1+\binom{1}{1}a^{1-1}b^1 = a+b$ ✓

$n=2$: $(a+b)^2 = \binom{2}{0}a^2+\binom{2}{1}a^1b^1+\binom{2}{2}a^0b^2 = a^2+2ab+b^2$ ✓

$n=3$: $(a+b)^3 = \binom{3}{0}a^3+\binom{3}{1}a^2b^1+\binom{3}{2}ab^2+\binom{3}{3}b^3 = a^3+3a^2b+3ab^2+b^3$ ✓

b)
$$\begin{aligned}
\binom{n}{k}+\binom{n}{k-1} &= \frac{n!}{k!(n-k)!}+\frac{n!}{(k-1)!(n-k+1)!} \\
&= \frac{n!(n-k+1)}{k!(n-k+1)!}+\frac{n!\,k}{k!(n-k+1)!} \\
&= \frac{n!(n-k+1)+n!\,k}{k!(n-k+1)!} \\
&= \frac{n!(n-k+1+k)}{k!(n-k+1)!} \\
&= \frac{n!(n+1)}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k} \text{ by defn.}
\end{aligned}$$

1.12 c) B.S. for $n=1$: $(a+b)^1$ checked in part a).

I.H. Suppose $n=k$, $P(k)$ is true, and

$$(a+b)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k$$

I.S. for $n=k+1$:

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)(a+b)^k \\
&= (a+b)\left(\binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \cdots + \binom{k}{k-1}ab^{k-1} + \binom{k}{k}b^k\right) \\
&= \binom{k}{0}a^{k+1} + \binom{k}{1}a^k b + \cdots + \binom{k}{k-1}a^2b^{k-1} + \binom{k}{k}ab^k + \\
&\quad \binom{k}{0}a^k b + \binom{k}{1}a^{k-1}b^2 + \cdots + \binom{k}{k-1}ab^k + \binom{k}{k}b^{k+1} \\
&= \binom{k}{0}a^{k+1} + \left(\binom{k}{0}+\binom{k}{1}\right)a^k b + \cdots + \left(\binom{k}{k-1}+\binom{k}{k}\right)ab^k + \binom{k}{k}b^{k+1} \\
&= \binom{k}{0}a^{k+1} + \binom{k+1}{1}a^k b + \cdots + \binom{k+1}{k}ab^k + \binom{k}{k}b^{k+1} \\
&= \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k b + \cdots + \binom{k+1}{k}ab^k + \binom{k+1}{k+1}b^{k+1}
\end{aligned}$$

2.1 Show $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{24}$, $\sqrt{31}$ are not rational numbers.

$\sqrt{3}$: by 2.3 Corollary, the only rational numbers that can be solns of $x^2-3=0$ are $\pm1, \pm3$.

Substituting $\pm1, \pm3$ into x^2-3 shows they're not solns.

Since $\sqrt{3}$ is a soln, it cannot be a rational number.

Similarly for $\sqrt{5}$, $\sqrt{7}$, $\sqrt{31}$, substituting $\{\pm1, \pm5\}$, $\{\pm1, \pm7\}$, $\{\pm1, \pm31\}$ respectively shows they're not solns. Hence $\sqrt{5}$, $\sqrt{7}$, $\sqrt{31}$ aren't rational numbers.

$\sqrt{24}$: $\sqrt{24} = 2\sqrt{6}$, suffice to show $\sqrt{6}$ is irrational.

possible solns for $x^2-6=0$: $\pm1, \pm2, \pm3, \pm6$, substituting shows they're not solns. Hence $\sqrt{24}$ isn't a rational number.

2.2 Show $\sqrt[3]{2}$, $\sqrt[7]{5}$, $\sqrt[4]{13}$ are not rational numbers.

$\sqrt[3]{2}$: $x^3-2=0$, possible soln: $\pm1, \pm2$

substituting shows they're not solns. Hence $\sqrt[3]{2}$ isn't a rational number.

by similar logic, $\sqrt[7]{5}$, $\sqrt[4]{13}$ are also not rational numbers.

2.7 Show they are rational:

a) $\sqrt{4+2\sqrt{3}}-\sqrt{3}$

$x = \sqrt{4+2\sqrt{3}}-\sqrt{3}$

$(x+\sqrt{3})^2 = 4+2\sqrt{3}$

$x^2+3+2\sqrt{3}x-4-2\sqrt{3}=0$

$x^2+2\sqrt{3}x-(1+2\sqrt{3})=0$

$(x-1)(1+2\sqrt{3})=0$

$x = \boxed{1 = \sqrt{4+2\sqrt{3}}-\sqrt{3}},$

So it's a rational number.

b) $\sqrt{6+4\sqrt{2}}-\sqrt{2}$

$= \sqrt{2+4\sqrt{2}+4}-\sqrt{2}$

$= \sqrt{(\sqrt{2}+2)^2}-\sqrt{2}$

$= (\sqrt{2}+2)-\sqrt{2}$

$= 2$

hence it's a rational number.

3.6. a) Prove $|a+b+c| \leq |a|+|b|+|c| \ \forall a, b, c \in \mathbb{R}$.

By triangle inequality, we know $|a+b| < |a|+|b| \ \forall a, b \in \mathbb{R}$.

Now consider $d = a+b$. then by triangle inequality,

$|d+c| \leq |d|+|c| \Rightarrow |a+b+c| \leq |a+b|+|c|$,

since $|a+b| < |a|+|b|$, $|a+b+c| \leq |a|+|b|+|c|$ $\square$

3.6 b) Induction: prove $|a_1+a_2+\dots+a_n| \leq |a_1|+|a_2|+\dots+|a_n|$ for n numbers $a_1, a_2, \dots, a_n$.

BS: $n=1$ $|a_1| \leq |a_1|$ ✓

IH: suppose its true for n : $|a_1+a_2+\dots+a_n| \leq |a_1|+|a_2|+\dots+|a_n|$

IS: for $n+1$: $|a_1+a_2+\dots+a_n+a_{n+1}| \leq |a_1+a_2+\dots+a_n|+|a_{n+1}|$,

and $|a_1+a_2+\dots+a_n|+|a_{n+1}| \leq \underbrace{|a_1|+|a_2|+\dots+|a_n|}_{\text{I.H.}}+|a_{n+1}|$,

hence $|a_1+a_2+\dots+a_n+a_{n+1}| \leq |a_1|+|a_2|+\dots+|a_n|+|a_{n+1}|$ $\square$

4.11 Consider $a, b \in \mathbb{R}$ where $a<b$. Use Denseness of $\mathbb{Q}$ in 4.7 to show there are infinitely many rationals between a and b.

4.7 $\mathbb{Q}$: $a, b \in \mathbb{R}$, $a<b$ $\Rightarrow$ there is a rational $r \in \mathbb{Q}$ s.t. $a<r<b$.

We know: $\exists\, m, n \in \mathbb{Z}$ s.t. $a<\frac{m}{n}<b$.

Suppose $\frac{m}{n}$ is the only rational between a and b.

then $an<m<bn$,

$\Rightarrow an-m<0<bn-m$.

$\because a, b \in \mathbb{R}$, and $n \in \mathbb{Z}$,

$\therefore an, bn \in \mathbb{R}$, and $an-m$, $bn-m \in \mathbb{R}$.

Statement means that in the set $\mathbb{R}$, $an-m$ is the lower bound and $bn-m$ is the upper bound which we know is false since $\mathbb{R}$ is an infinite set. So there are infinitely many m, n that satisfy $an-m<0<bn-m$ $\Rightarrow$ $\frac{m}{n}$ is not unique rational number s.t. $a<r<b$. $\square$

4.14 a) let $b \in B$. let $a \in A$.

then $a+b \in A+B$, and $a+b \leq \sup(A+B)$.

$a-b \leq \sup(A+B)-b$, since a was arbitrary,

$\sup(A+B)-b$ is an upper bound for A.

Then since $\sup A$ is the least upper bound for A, we must have

$\sup A \leq \sup(A+B)-b$.

$\Rightarrow$ $b \leq \sup(A+B)-\sup A$, $\Rightarrow$ $\sup(A+B)-\sup A$ is an upper bound for B.

so $\sup B \leq \sup(A+B)-\sup A$, So $\sup A+\sup B \leq \sup(A+B)$.

on the other hand, for any $c \in A+B$ $(c=a+b)$,

$c=a+b \leq \sup A+\sup B$.

$\because$ This shows $\sup A+\sup B$ is an upper bound for $A+B$, we must have

$\sup(A+B) \leq \sup A+\sup B$ as well.

Combine both directions $\square$.

4.14 b) let $c \in -(A+B)$, then $c=-(a+b)$ for some $a \in A$, $b \in B$,

so $c=(-a)+(-b) \in -A+(-B)$ On the other hand, if $c \in -A+(-B)$ then

$c = -a + -b$ some $a \in -A$ and $b \in -B$, so $c = -(a+b)$,

$a+b \in A+B$ meaning $c \in -(A+B)$.

Using this, $\inf(A+B) = -\sup(-(A+B)) = -\sup(-A + -B)$, by part a),

$-(\sup(-A) + \sup(-B)) = -\sup(-A) + -\sup(-B)$.

Then using $-\sup(-A) = \inf(A)$, $-\sup(-B) = \inf B$, $\square$.

7.5 a) $\lim s_n$, $s_n = \sqrt{n^2+1} - n = (\sqrt{n^2+1} - n)\dfrac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \dfrac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \dfrac{1}{\sqrt{n^2+1}+n}$

denominator $\sqrt{n^2+1}+n \to \infty$ as $n \to \infty$. so $s_n \to 0$.

b) $\lim \sqrt{n^2+n} - n$, $s_n = (\sqrt{n^2+n} - n)\dfrac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \dfrac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \dfrac{n}{\sqrt{n^2+n}+n}$

$= \dfrac{n}{n\sqrt{1+\frac{1}{n}}+n} = \dfrac{n}{n(\sqrt{1+\frac{1}{n}}+1)} = \dfrac{1}{\sqrt{1+\frac{1}{n}}+1}$

since $\sqrt{1+\frac{1}{n}} \to \infty$ as $n \to \infty$, $s_n \to 0$.

c) $\lim \sqrt{4n^2+n} - 2n$, $s_n = \sqrt{4n^2+n} - 2n = \sqrt{4n^2+n} - 2n \, \dfrac{\sqrt{4n^2+n}+2n}{\sqrt{4n^2+n}+2n}$

$= \dfrac{4n^2+n-4n^2}{\sqrt{4n^2+n}+2n} = \dfrac{n}{\sqrt{4n^2+n}+2n} = \dfrac{n}{n\sqrt{4+\frac{1}{n}}+2n} = \dfrac{n}{n(\sqrt{4+\frac{1}{n}}+2)} = \dfrac{1}{\sqrt{4+\frac{1}{n}}+2}$

Since $\sqrt{4+\frac{1}{n}} \to \infty$ as $n \to \infty$, $s_n \to 0$.