HW2

0. Discussion 9.9, 9.15, 10.7, 10.8

9.9 a) let $M>0$. Since $s_n \to +\infty$, there is $N_s \in \mathbb{N}$ s.t. $n > N_s$ implies $s_n > M$. Let $N = \max\{N_0, N_s\}$. If $n > N$, then $t_n = s_n > M$. So $t_n \to +\infty$.

b) let $M<0$. Since $s_n \to -\infty$, there is $N_s \in \mathbb{N}$ s.t. $n > N_s$ implies that $s_n < M$. Let $N = \max\{N_0, N_s\}$. If $n > N$, then $t_n = s_n < M$. So $t_n \to +\infty$.

c) Case 1: $\lim s_n$, $\lim t_n$ are finite, then $\lim s_n \le \lim t_n$.

Case 2: $\lim s_n$ not finite.

a) $\lim s_n = -\infty$. then $\lim s_n \le \lim t_n$ always holds b/c $\lim t_n$ takes values in $\mathbb{R} \cup \{+\infty, -\infty\}$, and for any $a \in \mathbb{R} \cup \{+\infty, -\infty\}$, $-\infty \le a$.

b) $\lim s_n = +\infty$. then $\lim t_n = +\infty$, $\lim s_n \le \lim t_n$ still holds.

Case 3: $\lim t_n$ not finite.

a) $\lim t_n = +\infty$, then $\lim s_n \le \lim t_n$ always holds b/c $\lim s_n$ takes values in $\mathbb{R} \cup \{+\infty, -\infty\}$, and for any $a \in \mathbb{R} \cup \{+\infty, -\infty\}$, $a \le +\infty$.

b) $\lim t_n = -\infty$. by b) $\lim s_n = -\infty$, so $\lim s_n \le \lim t_n$ still holds.

9.15 Show $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0 \quad \forall a \in \mathbb{R}$.

let $s_n = \frac{a^n}{n!}$ and we see that $\frac{s_{n+1}}{s_n} = \frac{a}{n+1}$ tends to zero as $n \to \infty$.

Hence $\lim s_n = 0$ ($n!$ grows faster than any exponential sequence a^n).

10.7 Since S is bounded, $\sup S \in \mathbb{R}$.

$\because$ $\sup S$ is the least upper bound of S, for any $n \in \mathbb{N}$, $\sup S - \frac{1}{n}$ is not an upper bound of S, and so there is an el't in S, denoted by s_n, which is greater than $\sup S - \frac{1}{n}$.

We obtain a sequence $(s_n) \in S$ s.t. $s_n > \sup S - \frac{1}{n}$ for any n.

$\because$ $\sup S$ is an upper bound of S and $s_n \in S$,

$\therefore$ $\sup S \ge s_n \ \forall n$. Applying Squeeze lemma to $\sup S \ge s_n > \sup S - \frac{1}{n}$, $s_n \to \sup S$.

10.8 Let s_n be an increasing sequence and define $\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}$.

$\because$ s_n is an increasing sequence,

$$\sigma_{n+1} - \sigma_n = \frac{n(s_1 + s_2 + \cdots + s_n + s_{n+1}) - (n+1)(s_1 + \cdots + s_n)}{n(n+1)}$$

$$= \frac{(s_{n+1} - s_1) + (s_{n+1} - s_2) + \cdots + (s_{n+1} - s_n)}{n(n+1)} \geq \frac{0 + 0 \cdots + 0}{n(n+1)} = 0$$, so σ_n is an increasing sequence.

1. Ross Ex 10.9, 10.10, 10.11

10.9 a) $s_1 = 1$, $s_2 = \frac{1}{2}$, $s_3 = \frac{1}{6}$, $s_4 = \frac{1}{48}$.

b) $s_{n+1} = \frac{n}{n+1} s_n^2 < s_n^2 < 1 \cdot s_n = s_n \quad \forall n \in \mathbb{N}$.

$s_{n+1} \leq s_n \ \forall n \in \mathbb{N}$, the sequence is monotonically non-increasing.

$\because$ $\{s_n\}$ is a bounded monotone sequence, it must converge.

c) let $s = \lim_{n \to \infty} s_n$. From the recursion relation,

$$\lim_{n \to \infty} s_{n+1} = \lim_{n \to \infty} \frac{n}{n+1} s_n^2 \Rightarrow s = s^2$$

$$s(s-1) = 0, \Rightarrow s = 0, 1$$

$\because$ $s_n \leq \frac{1}{2}$ for $n \geq 2$ and non-increasing, $s \leq \frac{1}{2} \Rightarrow s = 0$. $\square$.

10.10 a) $s_2 = \frac{1}{3}(1+1) = \frac{2}{3}$, $s_3 = \frac{1}{3}(\frac{2}{3}+1) = \frac{5}{9}$, $s_4 = \frac{1}{3}(\frac{5}{9}+1) = \frac{14}{27}$.

b) B.S. $s_1 = 1 > \frac{1}{2}$.

Suppose $s_n > \frac{1}{2}$, $s_{n+1} = \frac{1}{3}(s_n + 1) > \frac{1}{3}(\frac{1}{2}+1) = \frac{1}{2}$.

hence $s_n > \frac{1}{2}$ for all n.

c) M.M.I: B.S.: $s_1 \geq s_2$, since $s_1 = 1$, $s_2 = \frac{2}{3}$.

Suppose $s_n \geq s_{n+1}$. Then $s_{n+1} = \frac{1}{3}(s_n + 1) \geq \frac{1}{3}(s_{n+1} + 1) = s_{n+2}$.

Hence $s_n \geq s_{n+1} \ \forall n$. $\square$

d) Since (s_n) is decreasing and bounded below, it conv. to a real number, m.

$s_{n+1} = \frac{1}{3}(s_n + 1)$, by limit theorems, s_{n+1} conv. to $\frac{1}{3}(m+1)$.

Since $\lim s_{n+1} = \lim s_n$, $\frac{1}{3}(m+1) = m$. $\Rightarrow m = \frac{1}{2}$, $\lim s_n = \frac{1}{2}$. $\square$

10.11 a) (t_n) is a decreasing sequence since t_{n+1} is obtained by multiplying t_n by a fraction b/w 0 and 1.

(t_n) bounded below by 0. $\because$ it's decreasing, it's bounded above by $t_1 = 1$.

In section 10, it's shown that monotonic seq converge, so t_n must converge.

b) Guess: $\lim = \frac{2}{3}$,

express $t_{n+1} = \dfrac{(2n+1)!\,(2n-1)!}{2^{4n-1}(n!)^3(n-1)!}$ (might be correct?)

2. current thoughts:

$|a_n - L| < \epsilon \Rightarrow -\epsilon < a_n - L \le b_n - L$

$|c_n - L| < \epsilon \Rightarrow b_n - L \le c_n - L < \epsilon$

$\Rightarrow$

Squeeze test:

$\epsilon > 0$, $\exists M, N$ s.t. $\forall n > N$: $a_n > L - \epsilon$

and $\forall m > M$: $c_m < L + \epsilon$.

Then for any $n > \max(M, N)$:

$L - \epsilon < a_n \le b_n \le c_n < L + \epsilon$

$-\epsilon < b_n - L < \epsilon$

$|b_n - L| < \epsilon$.

$\Rightarrow \lim b_n = L$.