HW3 10.6, 11.2, 11.3, 11.5

10.6 a) $|s_{n+1} - s_n| < 2^{-n}$ $\forall n \in \mathbb{N}$. Prove (s_n) is a Cauchy seq and hence a conv. seq.

for $m, n \in \mathbb{N}$, $m>n$, <u>get only one "n"-index</u>:

$$|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - s_{m-2} + s_{m-2} + \dots - s_n|$$

$$\leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \dots + |s_{n-1} - s_n|$$

$$< \underbrace{\frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \dots + \frac{1}{2^n}}$$

$$\boxed{a(1+r+r^2+\dots+r^n) = a\left(\frac{1-r^{n+1}}{1-r}\right)}$$

$$\frac{1}{2^n}\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\dots+\left(\frac{1}{2}\right)^{m-1-n}\right) = \frac{1}{2^n}\left(\frac{1-\left(\frac{1}{2}\right)^{m-n}}{1-\frac{1}{2}}\right) = \frac{1}{2^{n-1}} - \frac{1}{2^{m-1}}$$

$$|s_m - s_n| < \frac{1}{2^{n-1}} - \frac{1}{2^{m-1}} < \frac{1}{2^{n-1}}$$

Since $|s_m - s_n| < \frac{1}{2^{n-1}}$ $\forall m, n > N$, and for $\frac{1}{2^{n-1}}$: $\lim_{n\to\infty} \frac{1}{2^{n-1}} = 0$ which means $\forall n$,

$\exists n > N$ s.t. $\left|\frac{1}{2^{n-1}} - 0\right| < \epsilon$, hence $|s_m - s_n| < \frac{1}{2^{n-1}} < \epsilon$. $\square$

10.6 b) $|s_{n+1} - s_n| < \frac{1}{n}$ $\forall n \in \mathbb{N}$.

No, counterexample: $s_n = \sqrt{n+1}$ $\quad |\sqrt{n+1} - \sqrt{n}| < \frac{1}{n}$, but $\lim_{n\to\infty} \sqrt{n+1} = \infty$, diverges.

by defn of Cauchy, this example fails.

11.2

① $a_n = (-1)^n$

a) $\{1, 1, 1, 1 \dots 1\}$ is monotone

b) $\{-1, 1\}$ is the subsequential limit

c) $\lim\sup a_n = 1$, $\lim\inf a_n = -1$

d) a_n does not conv. it oscillates b/w ± 1.

e) a_n is bounded. $a_n = \pm 1$.

② $b_n = \frac{1}{n}$

a) $n = 3k$: $\{\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \dots\}$ is monotone

b) $\{0\}$ is the subsequential limit

c) $\lim\sup b_n = 0$, $\lim\inf b_n = 0$

d) b_n conv. to 0

e) b_n is bounded between 0 and 1.

③ $c_n = n^2$

a) $\{0, 1, 4, 9, \dots\}$ is monotone

b) $\{\infty\}$ is the subsequential limit

c) $\lim\sup a_n = \infty$ $\quad \lim\inf a_n = \infty$

d) n^2 does not conv. it diverges.

e) n^2 is not bounded.

④ $d_n = \frac{6n+4}{7n-3}$

a) d_n itself is decreasing, monotone.

b) $\{\frac{6}{7}\}$ is the sequential limit

c) $\lim\sup a_n = \frac{6}{7}$ $\quad \lim\inf a_n = \frac{6}{7}$

d) $\frac{6n+4}{7n-3}$ conv. to 0.

e) d_n bounded: $[\frac{10}{4}, \frac{6}{7})$

11.3 ① $S_n = \cos\left(\frac{n\pi}{3}\right)$

a) when $n_k = 6k$, $\cos\left(\frac{6k\pi}{3}\right) = \cos(2k\pi) = 1$. ⇒ is monotone.

b) $\{0, \pm\sqrt{\frac{3}{2}}\}$ is the subsequential limits,

c) $\lim\sup s_n = \sqrt{\frac{3}{2}}$, $\lim\inf s_n = -\sqrt{\frac{3}{2}}$

d) S_n does not converge.

e) S_n is bounded by $[-1, 1]$

② $t_n = \frac{3}{4n+1}$

a) t_n itself is decreasing, so we can take it as its own subseq ⇒ is monotone.

b) $\{0\}$ is the subsequential limits

c) $\lim\sup\{t_n\} = 0$, $\lim\inf\{t_n\} = 0$

d) t_n converges to 0.

e) t_n is bounded $(0, \frac{3}{5}]$

③ $u_n = (-\frac{1}{2})^n$

a) $\{-\frac{1}{2}, -\frac{1}{8}, \ldots\}$ is decreasing

b) $\{0\}$ is the subsequential limit

c) $\lim\sup\{u_n\} = 0$, $\lim\inf\{u_n\} = 0$

d) u_n conv. to 0.

e) u_n bounded by $[-\frac{1}{2}, \frac{1}{4}]$.

④ $v_n = (-1)^n + \frac{1}{n}$

a) when $n = 2k$, v_n is decreasing ⇒ monotone

b) $\{-1, 1\}$ is the subsequential limit

c) $\lim\sup\{v_n\} = 1$, $\lim\inf\{v_n\} = -1$.

d) v_n does not converge.

e) v_n bounded by $(-1, \frac{3}{2}]$.

11.5 (q_n) be enumeration of all the rationals in the interval $(0,1]$.

a) Give the set of subsequential limits for (q_n).

Let S be the set of subsequential limits of (q_n).

$\because$ $x<0$, $x \notin S$, and $q_n > \frac{x}{2}$ $\forall n$,

$\therefore$ $s \in S$ must also satisfy $(s > x)/2 > x$.

Similarly, any $x > 1$ cannot be in S b/c for any such x, $q_n < \frac{1+x}{2}$ $\forall n$ so $s \leq \frac{1+x}{2}$ for any $s \in S$. $\Rightarrow S \subseteq [0,1]$.

Show $S = [0,1]$: let $x \in [0,1]$. Then take $n_1 = 1$ and define n_k inductively: take $n_{k+1} > n_k$ so that $q_{n_{k+1}} \in (0,1) \cap (\frac{x-1}{k}, \frac{x+1}{k})$. This is possible b/c for each k the intersection $(0,1) \cap (\frac{x-1}{k}, \frac{x+1}{k})$ is an interval of nonzero length, therefore it contains infinitely many rational numbers

$\Rightarrow$ q_n with $n < n_k$ cannot have exhausted all rational #s in the intersection,

So we must be able to find some $n_{k+1} > n_k$ with $q_{n_{k+1}} \in (0,1) \cap (\frac{x-1}{k}, \frac{x+1}{k})$.

Then the subseq (q_{n_k}) of (q_n) defined will conv. to x b/c for each k, $|q_{n_k} - x| < \frac{1}{k}$, so given $\epsilon > 0$, we can make $|q_{n_k} - x| < \epsilon$ by taking $k > \frac{1}{\epsilon}$.

This shows $[0,1] \subseteq S$ $\Rightarrow$ $S = [0,1]$.

b) Give the values of $\lim\sup q_n$ and $\lim\inf q_n$.

$\lim\sup q_n = \sup [0,1] = 1$, $\lim\inf q_n = \inf [0,1] = 0$.

2. Difference b/w limsup and sup? What is most counter-intuitive about limsup?
State some sentences that seems to be correct, but is actually wrong?

Sup: supremum of the actual seq.

limsup: supremum of the limit of the seq.

counter-intuitive about limsup: limsup $\neq$ sup.

statements that seems to be correct, but is actually wrong: limsup = sup.