HW4 Ross 12.10, 12.12, 14.2, 14.10 ; Rudin 3.6, 3.7, 3.9, 3.11

12.10 Prove (s_n) is bounded iff $\limsup |s_n| < +\infty$.

① Suppose (s_n) is bounded above. Then by defn, $\limsup |s_n| = t < +\infty$.

② Suppose $\limsup |s_n| < +\infty$. Then $\limsup |s_n| = t$.

$\because \lim(\sup\{|s_m| \text{ s.t. } m > n\}) = t$, $\exists N \in \mathbb{N}$ s.t.

$$|\sup\{|s_m| \text{ s.t. } m > N\} - t| < 1, \Rightarrow \sup\{|s_m| \text{ s.t. } m > N\} < t+1,$$

So for all $m > N$, $|s_m| < t+1$,

and for all n, $|s_n| \le \max\{|s_1|, |s_2|, \dots, |s_N|, t+1\}$. hence $(s_n)_{n\in\mathbb{N}}$ is bounded.

12.12 (s_n) be a seq of nonnegative numbers; for each n define

$\sigma_n = \frac{1}{n}(s_1 + s_2 + \dots + s_n)$

a) Show $\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n$.

c) $\lim \sigma_n$ exists, $\lim s_n$ DNE.

$s_n = (-1)^n$

$\lim \sigma_n = 0$,

$\lim s_n$ DNE

Prove $\limsup \sigma_n \le \limsup s_n$:

given M and N s.t. $M > N$, we claim that

$$\sup_{n>M} \sigma_n \le \frac{s_1 + \dots + s_N}{M} + \sup_{k>N} s_k \quad (2)$$

Proof (2): Show $\sigma_n \underset{n>M}{\le} \frac{s_1 + \dots + s_N}{M} + \sup_{k>N} s_k$ (3) for each $n > M$.

$\because n > M > N$, we break σ_n into two parts:

$$\sigma_n = \frac{s_1 + \dots + s_n}{n} = \frac{s_1 + \dots + s_N}{n} + \frac{s_{N+1} + \dots + s_n}{n} \quad (4)$$

$\because n > M$ { ① $\frac{s_1+\dots+s_N}{n}$ in (4) is less than $\frac{s_1+\dots+s_N}{M}$ in (3).

② $\frac{s_{N+1}+\dots+s_n}{n}$ in (4) is $\le$ than $\frac{s_{N+1}+\dots+s_N}{n}$ in (3).

$\Rightarrow$ Combine these two statements ① ②, we see that (2) holds.

Then fix N, we take the limit of (2) as $M \to \infty$, we see that

$\limsup \sigma_n \le \sup_{k>N} s_k$. (5)

Taking the limit of (5) as $N \to \infty$ $\limsup \sigma_n \le \limsup s_k$.

The proof of $\liminf s_n \le \liminf \sigma_n$ is similar.

b) Show if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.

if $\lim s_n$ exists, then $\liminf s_n = \limsup s_n$,

So $\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n$ becomes

$$\liminf s_n = \liminf \sigma_n = \limsup \sigma_n = \limsup s_n.$$

(if a seq in conv., its seq of average is also conv.)

14.2

a) $\sum \frac{n-1}{n^2}$, $\frac{n-1}{n^2} \sim \frac{1}{n}$, since $\sum \frac{1}{n}$ is div $\Rightarrow \sum \frac{n-1}{n^2}$ is div

alternatively: $\frac{n-1}{n^2} > c \cdot \frac{1}{n}$, for $n > 2$: $\frac{n-1}{n^2} > \frac{n - \frac{2}{n}}{n^2} = \frac{\frac{1}{2}n}{n^2} = \frac{1}{2} \cdot \frac{1}{n}$,
since $\sum \frac{1}{n}$ is div, $\Rightarrow \sum \frac{n-1}{n^2}$ is div.

b) $\sum (-1)^n$: $\sum (-1)^n$ is div b/c limit DNE for $(-1)^n$.

c) $\sum \frac{3n}{n^3} = \sum \frac{3}{n^2}$, $\sum \frac{3}{n^2} = 3 \sum \frac{1}{n^2}$, $\because \sum \frac{1}{n^2}$ conv., $\therefore 3 \sum \frac{1}{n^2} = \sum \frac{3n}{n^3}$ conv.

d) $\sum \frac{n^3}{3^n}$, $\lim_{n\to\infty} \frac{n^3}{3^n} = 0$ b/c 3^n is exponential in n, n^3 is polynomial in n. $(3^n = e^{\ln|3| \cdot n})$

e) $\sum \frac{n^2}{n!}$, $\lim_{n\to\infty} \left| \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{\cancel{2}}}{\cancel{(n+1)}\, n^2} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n^2} \right| = 0 < 1$, Conv. absolutely.

f) $\sum \frac{1}{n^n}$, $\limsup \left| \frac{1}{n^n} \right|^{\frac{1}{n}} = \limsup \left| \frac{1}{n} \right| = 0$, $\Rightarrow$ conv. absolutely.

e) $\limsup \left| \frac{a_{n+1}}{a_n} \right| = \liminf = 0$
$\Rightarrow \sum a_n$ conv.

g) $\sum \frac{n}{2^n}$ $\lim \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right| = \frac{1}{2} \lim \left| 1 + \frac{1}{n} \right|$
$= \frac{1}{2} < 1$
hence conv.

14.10

$\sum a_n$: diverges by the Root Test but for which the Ratio Test gives no information.

ratio test
no info: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$, $\liminf \left| \frac{a_{n+1}}{a_n} \right| \le 1 \le \limsup \left| \frac{a_{n+1}}{a_n} \right|$

root test
div: $\alpha = \limsup |a_n|^{\frac{1}{n}} > 1$.

Rudin 3.6, 3.7, 3.9, 3.11

3.6

a) $a_n = \sqrt{n+1} - \sqrt{n}$, $\quad a_n = \dfrac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \dfrac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}} > \dfrac{1}{2\sqrt{n+1}}$ which div. $(p=\frac{1}{2})$

$\therefore$ a_n div.

b) $a_n = \dfrac{\sqrt{n+1}-\sqrt{n}}{n}$ $\quad a_n = \dfrac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{n(\sqrt{n+1}+\sqrt{n})} = \dfrac{n+1-n}{n(\sqrt{n+1}+\sqrt{n})} = \dfrac{1}{n(\sqrt{n+1}+\sqrt{n})}$

$= \dfrac{1}{2n\sqrt{n+1}} < \dfrac{1}{n^{\frac{3}{2}}}$ which conv. $(p=\frac{3}{2})$, $\therefore$ b_n conv.

c) $a_n = (\sqrt[n]{n}-1)^n \cancel{= (n^{\frac{1}{n}}-1)^n > (n^{\frac{1}{2n}})^n}$ ~~which div. b/c~~ $\cancel{(n^{\frac{1}{2n}})^n = n^{\frac{1}{2}}}$ ~~div.~~

$\sqrt[n]{n} \to 1$ as $n \to \infty$, hence $a_n \to (1-1)^n \to 0$ as $n \to \infty$

Using root test, $a_n^{\frac{1}{n}} = \sqrt[n]{n} - 1 \to 0$. $\therefore$ a_n conv.

d) $a_n = \dfrac{1}{1+z^n}$ for complex values of z. $\quad |z| \leq 1$, then $a_n \geq \frac{1}{2}$ so it doesn't go to zero. $\Rightarrow$ div.

$|z| > 1$, then $r = \frac{1}{|z|} < 1$ which conv. by geometric series.

3.7. Prove conv. of $\sum a_n$ implies conv. of $\sum \frac{\sqrt{a_n}}{n}$. if $a_n \geq 0$.

$\sum a_n$ conv: then $\lim_{n\to\infty} a_n = 0$.

Now $\sum \frac{\sqrt{a_n}}{n}$: $\lim_{n\to\infty} \dfrac{\sqrt{a_n}}{n} = \dfrac{\lim_{n\to\infty}\sqrt{a_n}}{\lim_{n\to\infty} n} = 0$

$(\sqrt{a_n} - \frac{1}{n})^2 \geq 0$,

$(a_n)^2 + \frac{1}{n^2} - \frac{2\sqrt{a_n}}{n} \geq 0$

$\frac{\sqrt{a_n}}{n} \leq \frac{1}{2}\left((a_n)^2 + \frac{1}{n^2}\right)$

$\because$ $\sum a_n$ conv, $\sum (a_n)^2$ conv. b/c $a_n < 1$ for $n \to \infty$, $\Rightarrow (a_n)^2 < a_n$.

$\sum \frac{1}{n^2}$ conv. $(p=2)$,

$\therefore$ $\frac{\sqrt{a_n}}{n}$ conv.

$\sum c_n z^n$, $\alpha = \lim_{\infty} \sup \sqrt[n]{|c_n|}$, $R = \frac{1}{\alpha}$ $\begin{cases} \alpha = 0, & R = +\infty \\ \alpha = +\infty, & R = 0. \\ \text{conv: } & |z| < R \\ \text{div: } & |z| > R \end{cases}$

3.9 Find the radius of convergence of each of the following series:

a) $\sum n^3 z^n$, $a_n = n^3$, $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = 1$, $R=1$

b) $\sum \frac{2^n}{n!} z^n$, $a_n = \frac{2^n}{n!}$, $\lim \frac{a_n}{a_{n+1}} = \frac{2^n}{n!} \times \frac{(n+1)!}{2^{n+1}} = \lim_{n\to\infty} \frac{n+1}{2} = \frac{1}{2} \lim_{n\to\infty} n+1 = \infty$, $R=\infty$.

c) $\sum \frac{2^n}{n^2} z^n$, $a_n = \frac{2^n}{n^2}$ $\lim_{\infty} \frac{a_n}{a_{n+1}} = \frac{2^n}{n^2} \times \frac{(n+1)^2}{2^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2} = \frac{1}{2} \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \frac{1}{2}$, $R = \frac{1}{2}$

d) $\sum \frac{n^3}{3^n} z^n$, $a_n = \frac{n^3}{3^n}$, $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{n^3}{3^n} \cdot \frac{3^{n+1}}{(n+1)^3} = \lim_{n\to\infty} \frac{3n^3}{(n+1)^3} = 3 \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 = 3$, $R=3$

3.11 $a_n > 0$, $S_n = a_1 + \dots + a_n$, $\sum a_n$ div.

a) Prove that $\sum \frac{a_n}{1+a_n}$ div.

$\cancel{\sum \frac{a_n}{1+a_n} = \sum \frac{1+a_n-1}{1+a_n} = \sum\left(1 - \frac{1}{1+a_n}\right), \quad \lim_{n\to\infty} 1 - \frac{1}{1+a_n} \to 1 \neq 0, \text{ so } \sum a_n \text{ div.}}$

Proof by contradiction: Suppose $\sum \frac{a_n}{1+a_n}$ is conv, then $\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0$.

$\lim_{n\to\infty} \frac{a_n}{1+a_n} = \lim_{n\to\infty} 1 - \frac{1}{1+a_n} = 0$, $\Rightarrow \lim_{n\to\infty} \frac{1}{1+a_n} = 1 \Rightarrow a_n \to 0$.

this is a contradiction since $\sum a_n$ div. $\square$

b) Prove $\frac{a_{N+1}}{S_{N+1}} + \dots + \frac{a_{N+k}}{S_{N+k}} \geq 1 - \frac{S_N}{S_{N+k}}$, deduce that $\sum \frac{a_n}{S_n}$ div.

$\frac{a_{N+1}}{S_{N+1}} + \dots + \frac{a_{N+k}}{S_{N+k}} \geq \frac{1}{S_{N+k}} (a_{N+1} + a_{N+2} + \dots + a_{N+k})$

$= \frac{1}{S_{N+k}} (S_{N+k} - S_N)$ $\quad (S_{N+k} = a_1 + \dots + a_n + \underbrace{a_{n+1} + a_{n+2} + \dots + a_{n+k}})$

$= 1 - \frac{S_N}{S_{N+k}}$

c) Prove $\frac{a_n}{S^2_n} \leq \frac{1}{S_{n-1}} - \frac{1}{S_n}$ and deduce that $\sum \frac{a_n}{S^2_n}$ conv.

RHS $= \frac{S_n - S_{n-1}}{(S_{n-1})S_n} = \frac{a_n}{(S_{n-1})S_n}$

NTS: $S^2_n \geq (S_{n-1})S_n$,

since $a_n > 0$, $S_n > S_{n-1}$, so $S^2_n \geq (S_{n-1})S_n \Rightarrow \frac{a_n}{S^2_n} \leq \frac{a_n}{(S_{n-1})S_n}$

$\lim_{n\to\infty} \sum \frac{1}{S_{n-1}} - \frac{1}{S_n} = \frac{1}{a_n}$, conv. to $\frac{1}{a_n}$, by comparison test, $\sum \frac{a_n}{S^2_n}$ conv.

d) $\sum \frac{a_n}{1+na_n}$ and $\sum \frac{a_n}{1+n^2 a_n}$

$\downarrow$ conv. or div. $\qquad \downarrow$ converges since $p=2$.

if $a_n = \frac{1}{n^2}$, $\sum \frac{a_n}{1+na_n}$ conv.

if $na_n < k$, then $\sum \frac{a_n}{1+k} = \sum a_n \cdot \frac{1}{1+k}$ which div.