HW7

1. If X and Y are open cover compact, prove that $X \times Y$ is open cover compact.

We know X and Y containe finite subcovers.

let K be a closed subset of X, M be a closed subset of Y.

let $(k_n, m_n) \in K \times Y \in X \times Y$. Since X and Y are open cover compact,

then $\exists \{G_\alpha : \alpha \in I\}$ of open subsets of X s.t. $K \subseteq \bigcup_{\alpha \in I} G_\alpha$,

and $\exists \{H_\beta : \beta \in I\}$ of open subsets of Y s.t. $M \subseteq \bigcup_{\beta \in I} H_\beta$.

$\because k_n \subset K$ and $m_n \subset M$,

$\therefore k_n \in \bigcup_{\alpha \in I} G_\alpha$ and $m_n \in \bigcup_{\beta \in I} N_\beta$ for some $\alpha = n$ and $\beta = n$.

$(k_n, m_n) = (k, m)_n = (G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}) \cup (N_{\beta_1} \cup N_{\beta_2} \cup \dots \cup N_{\beta_n}) = K_n \cup M_n$ for some n.

hence for any subset $(k_n, m_n) \in X \times Y$, there exists $\bigcup_{\alpha, \beta \in I} (G_\alpha \cup N_\beta)$ s.t.

$(k_n, m_n) \subseteq \bigcup_{\alpha, \beta \in I} (G_\alpha \cup N_\beta)$. $\square$

2. $f: X \to Y$ is a continuous map b/w metric spaces. $A \subset X$ ← proper subset

False a) A open $\Rightarrow$ $f(A)$ open.

counterex: $f: x \mapsto x^2$ is continuous, but it sends the open interval $(-1, 1)$ to $[0, 1)$ which is not open.

True b) A closed $\Rightarrow$ $f(A)$ closed

question: is $f(\underline{\text{limit}}) \subset f(A)$?

Since limit $\in A$, $f(\text{limit} \in A) \in f(A)$ ☺

False c) A bounded $\Rightarrow$ $f(A)$ bounded

$(0, 1) \xrightarrow{f = \frac{1}{x}} (1, \infty)$ which is not bounded.

True d) A compact $\Rightarrow$ $f(A)$ compact

Every seq $(a_{n_k}) \in A$ has a convergent subseq,

Since f is continuous, each convergent subseq still converges in $f(A)$

$\Rightarrow f(A)$ compact.

True e) A connected $\Rightarrow$ $f(A)$ connected

By thm 43 in Pugh (p87), the continuous image of a connected subset is connected.

$(0,1) \to \mathbb{R}$ tho :0

(there is a surjective map from)

3. Prove: there is no continuous map $f: [0,1] \to \mathbb{R}$ s.t. f is surjective.

<u>Pf</u> Sps f is continuous, and that f is surjective.

Since $[0,1]$ is compact, by exercise 2, $f([0,1]) = \mathbb{R}$ is compact.

We know $\mathbb{R}$ is <u>not</u> compact. $\Rightarrow\!\Leftarrow$ Hence proved. $\square$