HW5 13.3, 13.5, 13.7, 4, 5

13.3 $B = \{x \mid x = (x_1, x_2, \dots)\}$. $d(x,y) = \sup\{|x_j - y_j| : j = 1, 2, \dots\}$.

a) Show d is a metric for B.

① $d(x,x) = \sup\{|x_j - x_j| : j=1,2,\dots\} = 0$ ✓

② $d(y,x) = \sup\{|y_j - x_j| : j=1,2,\dots\} = \sup\{|x_j - y_j| : j=1,2,\dots\} = d(x,y)$ ✓

③ $d(x,z) = \sup\{|x_j - z_j| : j=1,2,\dots\}$

$d(x,y) = \sup\{|x_j - y_j| : j=1,2,\dots\}$ $\because |x_j - z_j| \le |x_j - y_j| + |y_j - z_j|$ by triangle inequality,

$d(y,z) = \sup\{|y_j - z_j| : j=1,2,\dots\}$ $\therefore \sup|x_j - z_j| \le \sup(|x_j - y_j| + |y_j - z_j|)$.

$d(x,z) \le d(x,y) + d(y,z)$

b) $d^*(x,y) = \sum_{j=1}^{\infty} |x_j - y_j| \overset{?}{\to}$ define a metric for B

The sequences are bounded, series $\sum_{i=1}^{\infty} |x_i - y_i|$ need not converge.

$\Rightarrow d^*$ is not a metric defined on all pairs.

Counterex: $x = (1, 1, \dots, 1)$, $y = (0, 0, \dots, 0)$, $\sum_{i=1}^{\infty} |x_i - y_i| = \sum_{i=1}^{\infty} 1$ doesn't conv.

13.5 $A \setminus B$ relative complement (objects that belong to A but not to B).

a) Verify $\bigcap\{S \setminus U : U \in \mathcal{U}\} = S \setminus \bigcup\{U : U \in \mathcal{U}\}$

let $x \in \bigcap\{S \setminus U : U \in \mathcal{U}\}$ and $y \in S \setminus \bigcup\{U : U \in \mathcal{U}\}$

① Show $x \in S \setminus \bigcup\{U : U \in \mathcal{U}\}$:

$\Rightarrow x \in S \setminus U$ for every $U \in \{U\}$,

$\Rightarrow x \notin U$ for all $U \in \mathcal{U}$

$\Rightarrow x \notin \bigcup\{U : U \in \mathcal{U}\}$

$\Rightarrow x \in S \setminus \bigcup\{U : U \in \mathcal{U}\}$

② show $y \in \bigcap\{S \setminus U : U \in \mathcal{U}\}$

$\Rightarrow$ we know $y \in S \setminus \bigcup\{U : U \in \mathcal{U}\}$

$\Rightarrow y \notin \bigcup\{U : U \in \mathcal{U}\}$

$\Rightarrow$ for all $U \in \mathcal{U}$, $y \notin U$, $y \in S \setminus U$.

$\Rightarrow y \in \bigcap\{S \setminus U : U \in \mathcal{U}\}$.

b) Show that the intersection of any collection of closed sets is a closed set

Pf. Suppose $\mathcal{E} = E_1, E_2, \dots, E_n \subseteq S$ are all closed.

then $S \setminus \mathcal{E}$ is open for every $E \in \mathcal{E}$ b/c $S \setminus \mathcal{E}$ is the relative complement.

take $\mathcal{U} = \{S \setminus E : E \in \mathcal{E}\}$ which is a family of open sets,

and $\mathcal{E} = \{S \setminus U : U \in \mathcal{U}\}$ b/c for each $E \in \mathcal{E}$, $E = S \setminus (S \setminus E)$

then $\bigcap\{E \in \mathcal{E}\} = \bigcap\{S \setminus U : U \in \mathcal{U}\} = S \setminus \bigcup\{U : U \in \mathcal{U}\}$ by De Morgan's law.

$\Rightarrow \bigcap\{E \in \mathcal{E}\} = S \setminus \bigcup\{U : U \in \mathcal{U}\}$

taking complements of both sides

$\Rightarrow S \setminus \bigcap\{E \in \mathcal{E}\} = \underbrace{\bigcup\{U : U \in \mathcal{U}\}}_{\text{open}}$

CONTD $\longrightarrow$

$\Rightarrow \quad S \setminus \bigcap \{E \in \mathcal{E}\} = \underbrace{\bigcup \{U : U \in \mathcal{U}\}}_{\text{open}}$

this means that the complement of $\bigcap\{E \in \mathcal{E}\}$ is open.

$\Rightarrow \bigcap\{E \in \mathcal{E}\}$ is closed. $\square$.

13.7 Show that every open set in $\mathbb{R}$ is the disjoint union of a finite or infinite seq of open intervals.

$\Rightarrow$ NTS: any open subset $S \subseteq \mathbb{R}$ can be expressed as the union $U = \bigcup_{i \in U} (a_i, b_i)$ of disjoint open intervals (allowing $\pm\infty$ as end points).

The completeness of the real numbers: on the extended real line $\mathbb{R} \cup \{+\infty\}$, every subset has a supremum and an infimum.

For each $x \in U$, let $a_x = \inf\{z < x \mid (z, x) \subset U\}$ and $b_x = \inf\{z > x \mid (x, z) \subset U\}$, then $I_x := (a_x, b_x)$ is a maximal open interval contained in U.

Note: for $x, y \in U$, the intervals I_x and I_y are either equal or disjoint; taking $I \subset U$ to be a set of representatives for the equivalence relation $x \sim y \Leftrightarrow I_x = I_y$, $\Rightarrow U = \bigcup_{x \in I} I_x$. $\square$

4. For a subset S of a metric space, prove that if $S_1 = \overline{S}$ and $S_2 = \overline{S}$, then $S_1 = S_2$.

$\overline{S}$: closure of S : $\{p \in X \mid$ there is a subseq (p_n) in S that converges to $p\}$

Want to show: $\overline{\overline{S}} = \overline{S}$.

Since $\overline{S} \subset \overline{\overline{S}}$, we need to show $\overline{\overline{S}} \subset \overline{S}$.

5. Prove that $\overline{S}$ is the intersection of all closed subsets in X that contains S. (may assume $\overline{S}$ is closed).