1.10 Basis case: $n=1$

$$2(1)+1 = 4(1)-1 = 3(1)^2 = 3$$

Assume

$$(2n+1)+(2n+3)+\dots+(4n-1) = 3n^2$$

There are n terms. $4n-1 - (2n+1) = 2n-2$
$n-1$ operations adding 2 are required to reach the nth term.

$$(2(n+1)+1)+(2(n+1)+3)+\dots+(4(n+1)-1)$$

Sequence has $n+1$ terms. The first n terms amount to: $(2n+1)+(2n+3)+\dots+(4(n)-1)+2n$
By the inductive hypothesis:

$$= 3n^2+2n+4(n+1)-1$$
$$= 3n^2+6n+3 = 3(n+1)^2$$

Thus P_{n+1} is true for P_n,
and by induction, $P(n)$ is true for all positive integers n.

1.12

(a) $n=1$:

$$(a+b)^1 = a+b$$

$n=2$:

$$(a+b)^2 = a^2+2ab+b^2$$

$n=3$:

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

(b)

$$\frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-k+1)!} \stackrel{?}{=} \frac{(n+1)!}{k!\,(n-k+1)!}$$

$$\frac{n!\,(n-k+1)}{k!\,(n-k+1)!} + \frac{n!\,k}{k!\,(n-k+1)!}$$

$$= \frac{n!\,(n+1+k-k)}{k!\,(n-k+1)!} = \frac{(n+1)!}{k(n-k+1)!}$$

(c) Basis case part A.

Assume the binomial theorem for n

Now: $(a+b)^{n+1} =$

$$(a+b)(a+b)^n = (a+b)\left[\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \dots \binom{n}{n}b^n\right]$$

$$= \left(\binom{n}{0}a^{n+1} + \binom{n}{1}a^nb + \dots \binom{n}{n}ab^n\right) + \left(\binom{n}{0}a^nb + \binom{n}{1}a^{n-1}b^2 + \dots + \binom{n}{n}b^{n+1}\right)$$

note $\binom{n}{0} = \binom{n+1}{0} = \binom{n}{n} = \binom{n+1}{n+1} = 1$ can be added as shown in b

$$= \binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^nb + \binom{n+1}{2}a^{n-1}b^2 + \dots + \binom{n+1}{n}ab^n + \binom{n+1}{n+1}b^{n+1}$$

2.1

$x^2 - 3 = 0$ can only have rational solutions ± 1, ± 3 neither work

$x^2 - 5 = 0$ ———— ± 1 ± 5, neither work,

$x^2 - 7 = 0$ ———— ± 1, ± 7, neither work

$x^2 - 24 = 0$ ———— ± 1, ± 2, ± 3, ± 4, ± 6, ± 8, ± 12 ± 24, none work

$x^2 - 31 = 0$ ———— ± 1, ± 31, neither work

2.2

$x^3 - 2 = 0$ ———— ± 1, ± 2

$x^7 - 5 = 0$ ———— ± 1 ± 5

$x^4 - 13$ ———— ± 1 ± 13

none work

2.7

(a) $a = \sqrt{4 + 2\sqrt{3}} - \sqrt{3}$ $\boxed{= 1}$

$a + \sqrt{3} = \sqrt{4 + 2\sqrt{3}}$

$a^2 + 2\sqrt{3}a + 3 = 4 + 2\sqrt{3}$

$a^2 + 2\sqrt{3}(a - 1) = 1$ $a = 1$

(b)

$a = \sqrt{6 + 4\sqrt{2}} - \sqrt{2}$ $\boxed{= 2}$

$a^2 + 2a\sqrt{2} + 2 = 6 + 4\sqrt{2}$

$a^2 + \sqrt{2}(2a - 4) = 4$ $a = 2$

3.6 (a) $|a+b+c|$

Let $d=a+b$ $|d+c|$

by the triangle inequality $|d+c| \leq |d|+|c|$

rewrite d in a and b $|a+b|+|c|$

by the triangle inequality $|a+b|+|c| \leq |a|+|b|+|c|$

Thus $|a+b+c| \leq |a|+|b|+|c|$

(b) $n=1$ obvious

$n=2$ shown in text

$n=3$ shown in (a)

Assume $|a_1+a_2+\dots+a_n| \leq |a_1|+|a_2|+\dots+|a_n|$

let $a_1+a_2+\dots+a_n = A$, thus $|A| \leq |a_1|+|a_2|+\dots+|a_n|$

consider for $n+1$ terms:

$$|a_1+a_2+\dots+a_n+a_{n+1}|$$

$$= |A+a_{n+1}|$$

Apply triangle inequality: $|A+a_{n+1}| \leq |A|+|a_{n+1}|$

Apply inductive hypothesis:

$$|A+a_{n+1}| \leq |a_1|+|a_2|+\dots+|a_n|+|a_{n+1}|$$

Expand A:

$$|a_1+a_2+\dots+a_n+a_{n+1}| \leq |a_1|+|a_2|+\dots+|a_n|+|a_{n+1}|$$

Thus the inductive hypothesis is true, by induction, for all n.

4.11

The denseness of $\mathbb{Q}$ shows that between any a and b in $\mathbb{R}$ where $a < b$ there is a rational r such that $a < r < b$.
Note that $r \in \mathbb{R}$. Thus, the denseness of $\mathbb{Q}$ shows that between r and b there is a rational, r', such that $r < r' < b$.
This process repeats infinitely without issue; thus there are infinite rationals between any two real numbers a and b with $a < b$.

4.14

(a) For all $a \in A$, $a \leq \sup(A)$
for all $b \in B$, $b \leq \sup(B)$

For all $a+b \in A+B$, $a+b \leq \sup(A) + \sup(B)$
Therefore the lowest upper bound of $a+b$ must be less than or equal to $\sup(A)+\sup(B)$
$\sup(A+B) \leq \sup(A) + \sup(B)$
let A' be A, with $\sup(A)$ added if not present. Likewise for B'.
Then $\sup(A') = \sup(A)$, $\sup(B') = \sup(B)$. Then, $\sup(A'+B') \geq \sup(A)+\sup(B)$. Then $\sup(A+B) = \sup(A)+\sup(B)$

(b) $\inf(S) = -\sup(-S)$ $\therefore \inf(A+B) = -\sup(-A-B)$
by 4.14 (a), $\inf(A+B) = -\sup(-A) + -\sup(-B)$
$= \inf(A) + \inf(B)$

2.5

(a) $\lim_{n\to\infty} \sqrt{n^2+1} - n = 0$

(b) $\dfrac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n)}{\sqrt{n^2+n} + n}$

$$\frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n}$$

$$= \frac{1}{\sqrt{1+1/n}+1}$$

$$\lim \sqrt{n^2+n} - n = 1/2$$

(c) $\dfrac{(\sqrt{4n^2+n} - 2n)(\sqrt{4n^2+n} + 2n)}{\sqrt{4n^2+n} + 2n}$

$$= \frac{4n^2+n-4n^2}{\sqrt{4n^2+n}+2n} = \frac{n}{\sqrt{4n^2+n}+2n}$$

$$= \frac{1}{\sqrt{4+1/n}+2}$$

$$\lim_{n\to\infty} \sqrt{4n^2+n} - 2n = \frac{1}{4}$$