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hw1

1.10
Define P_n to be the proposition that $(2n+1)+(2n+3)+...+(4n-1)=3n^2$, where $n \in \mathbb{N}$.
Suppose P_n is true, for some n.

$$\begin{aligned}
&(2(n+1)+1)+(2(n+1)+3)+...+(4(n+1)-1)\\
&=(2n+3)+(2n+5)+...+(4n+3)\\
&=-(2n+1)+(2n+1)+(2n+3)+(2n+5)+...+(4n-1)+(4n+1)+(4n+3)\\
&=-(2n+1)+3n^2+(4n+1)+(4n+3)\\
&=3n^2-2n-1+4n+1+4n+3\\
&=3n^2+6n+3\\
&=3(n^2+2n+1)\\
&=3(n+1)^2
\end{aligned}$$

So we proved that P_{n+1} is true, assuming P_n is true. Now we prove P_n is true for $n=1$, then all other n's will follow.
When $n=1$, $2n+1=3=3\cdot 1^2$. QED.

1.12.

(a)
- n = 1 : $(a+b)^1=\binom{1}{0}a^1+\binom{1}{n}b^1=a+b$
- n = 2: $(a+b)^2=a^2+2ab+b^2=\binom{2}{0}a^2+\binom{2}{1}a^1b^1+\binom{2}{2}b^2$
- n =3 :

$$\begin{aligned}
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\
&=\binom{3}{0}a^3+\binom{3}{1}a^2b+\binom{3}{2}ab^2+\binom{3}{3}b^3
\end{aligned}$$

(b)

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\
&= \frac{n!}{k!(n+1-k)!/(n+1-k)} + \frac{n!}{\frac{k!}{k}(n+1-k)!} \\
&= \frac{n!(n+1-k)}{k!(n+1-k)!} + \frac{n!k}{k!(n+1-k)!} \\
&= \frac{n!(n+1) - n!k + n!k}{k!(n+1-k)!} \\
&= \frac{(n+1)!}{k!(n+1-k)!} \\
&= \binom{n+1}{k}
\end{aligned}$$

(c) Let $n \in \mathbb{N}$, and define P_n to be the following proposition:
$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$
Suppose P_n is true. Then

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b) \cdot [\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{n}b^n]) \\
&= [\binom{n}{0}a^{n+1} + \binom{n}{1}a^n b + \binom{n}{2}a^{n-1}b^2 + ... + \binom{n}{n}ab^n] \\
&\qquad + [\binom{n}{0}a^n b + \binom{n}{1}a^{n-1}b^2 + ... + \binom{n}{n}b^{n+1}] \\
&= \binom{n}{0}a^{n+1} + \left[\binom{n}{0} + \binom{n}{1}\right]a^n b + ... + \binom{n}{n}b^{n+1} \\
&= \binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^n b + ... + \binom{n+1}{n+1}b^{n+1} \text{ (note that nC0 equals (n+1)C0)}
\end{aligned}$$

So we see that P_{n+1} follows from P_n. Since we proved in part a that P_1 is true, we conclude P_n(a.k.a the binomial theorem) is true for all non-zero natural number n. QED.

2.1

- $\sqrt{3}$ satisfies the equation $x^2 - 3 = 0$
  Divisors of -3: $\pm 1, \pm 3$
  Divisors of 1: ± 1
  Possible rational solutions according to Rational Zeros Theorem: $\frac{\pm 1}{\pm 1}, \frac{\pm 3}{\pm 1}$
  None of these equals $\sqrt{3}$, therefore $\sqrt{3}$ is not a rational number.

- $\sqrt{5}$ satisfies the equation $x^2 - 5 = 0$
  Divisors of -5: $\pm 1, \pm 5$
  Divisors of 1: ± 1
  Possible rational solutions according to Rational Zeros Theorem: $\frac{\pm 1}{\pm 1}, \frac{\pm 5}{\pm 1}$
  None of these equals $\sqrt{5}$, therefore $\sqrt{5}$ is not a rational number.

- $\sqrt{7}$ satisfies the equation $x^2 - 7 = 0$
  Divisors of -7: $\pm 1, \pm 7$
  Divisors of 1: ± 1
  Possible rational solutions according to Rational Zeros Theorem: $\frac{\pm 1}{\pm 1}, \frac{\pm 7}{\pm 1}$
  None of these equals $\sqrt{7}$, therefore $\sqrt{7}$ is not a rational number.

- $\sqrt{24}$ satisfies the equation $x^2 - 24 = 0$
  Divisors of -24: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$
  Divisors of 1: ± 1
  Possible rational solutions according to Rational Zeros Theorem: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$
  None of these equals $\sqrt{24}$, therefore $\sqrt{24}$ is not a rational number.

- $\sqrt{31}$ satisfies the equation $x^2 - 31 = 0$
  Divisors of -31: $\pm 1, \pm 31$
  Divisors of 1: ± 1
  Possible rational solutions according to Rational Zeros Theorem: $\pm 1, \pm 31$
  None of these equals $\sqrt{31}$, therefore $\sqrt{31}$ is not a rational number.

2.2

- $\sqrt[3]{2}$ satisfies the equation $x^3 - 2 = 0$
  Divisors of -2: $\pm 1, \pm 2$
  Divisors of 1: ± 1
  Possible rational solutions according to Rational Zeros Theorem: $\pm 1, \pm 2$
  None of these equals $\sqrt[3]{2}$, therefore $\sqrt[3]{2}$ is not a rational number.

- $\sqrt[7]{5}$ satisfies the equation $x^7 - 5 = 0$
  Divisors of -5: $\pm 1, \pm 5$
  Divisors of 1: ± 1
  Possible rational solutions according to Rational Zeros Theorem: $\pm 1, \pm 5$
  None of these equals $\sqrt[7]{5}$, therefore $\sqrt[7]{5}$ is not a rational number.

- $\sqrt[4]{13}$ satisfies the equation $x^4 - 13 = 0$
  Divisors of -13: $\pm 1, \pm 13$
  Divisors of 1: ± 1
  Possible rational solutions according to Rational Zeros Theorem: $\pm 1, \pm 13$
  None of these equals $\sqrt[4]{13}$, therefore $\sqrt[4]{13}$ is not a rational number.

2.7

(a) Define $x = \sqrt{4 + 2\sqrt{3}} - \sqrt{3}$

$$(x + \sqrt{3})^2 = x^2 + 3 + 2\sqrt{3}x = 4 + 2\sqrt{3}$$
$$x^2 + 2\sqrt{3}x = 1 + 2\sqrt{3}$$
$$x^2 - 1 = 2\sqrt{3} - 2\sqrt{3}x$$
$$(x + 1)(x - 1) = -2\sqrt{3}(x - 1)$$

Before we divide out the $(x - 1)$ term, we need to make sure $x \neq 1$. So let's try $x = 1$ and see where that leads us.
Suppose $x = 1$, then

$$1 = \sqrt{4 + 2\sqrt{3}} - \sqrt{3}$$
$$1 + \sqrt{3} = \sqrt{4 + 2\sqrt{3}}$$
$$1 + 3 + 2\sqrt{3} = 4 + 2\sqrt{3} \text{ (after squaring both sides)}$$
$$4 = 4$$

As shown above, $x = 1$ works. Therefore, recalling the definition of x, $x = \sqrt{4 + 2\sqrt{3}} - \sqrt{3} = 1 \in \mathbb{Q}$.

(b) Define $x = \sqrt{6 + 4\sqrt{2}} - \sqrt{2}$, then we have $x + \sqrt{2} = \sqrt{6 + 4\sqrt{2}}$. Similarly to part (a), it is easy to find that $x = 2$.

3.6

(a)

$$|a + b + c| = |(a + b) + c| \text{ (by associativity of addition)}$$
$$\leq |a + b| + |c| \text{ (by triangle inequality)}$$
$$\leq |a| + |b| + |c| \text{ (by triangle inequality)}$$

(b) Let P_n be the proposition that $|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$, where $a_i \in \mathbb{R}$ for $i = 1, 2, ..., n$.
Suppose P_n is true, then

$$|a_1 + a_2 + ... + a_n + a_{n+1}| = |(a_1 + a_2 + ... + a_n) + a_{n+1}| \text{ (by associativity of addition)}$$
$$\leq |a_1 + a_2 + ... + a_n| + |a_{n+1}| \text{ (by triangle equality)}$$
$$\leq |a_1| + |a_2| + ... + |a_n| + |a_{n+1}| \text{ (by the assumption)}$$

i.e we have shown that if P_n holds, then P_{n+1} also holds. Now we show the case for when n=1, then the rest will follow.
Suppose n = 1. $|a_1| \leq |a_1|$. (proof by duh)

4.11

Suppose there are N rationals between a and b, where $N \in \{1, 2, 3, ...\}$.
$\mathbb{R}$ can be ordered(Ross Chapter 3, property O1), so we can write the N rational numbers as $\{r_1, r_2, ..., r_N\}$, where $a < r_1 < r_2 < ... < r_N < b$.
Since $r_N \in \mathbb{Q}$ and $\mathbb{Q} \subset \mathbb{R}$, $r_N \in \mathbb{R}$.
Since $r_N \in \mathbb{R}$ and $r_N < b$, by Denseness of $\mathbb{Q}$ theorem there must exist another rational number, call it r_{N+1}, such that $r_N < r_{N+1} < b$. This is in contradiction with the assumption we started with, which is that there are only N rationals between a and b. So the assumption is nonsense, which means there are infinitely many rationals between a and b.

4.14

(a)

(≤)

$\forall a \in A, b \in B : a \leq sup(A), b \leq sup(B)$ by definition of sup

So $a + b \leq sup(A) + sup(B)$ for any a from A and b from B

So $sup(A) + sup(B)$ is an upper bound of $A + B$, by definition of upper bound

So $sup(A + B) \leq sup(A) + sup(B)$ by definition of sup (1)

(≥)

$\forall a \in A, b \in B : a + b \leq sup(A + B)$ by definition of A+B and definition of sup

Therefore $a \leq sup(A + B) - b$ for any a from A and b from B

Therefore $sup(A + B) - b$ is an upper bound of A, by definition

Therefore $sup(A) \leq sup(A + B) - b$ by definition of sup

Therefore $b \leq sup(A + B) - sup(A)$

Since $b \in B$ was arbitrary, $sup(A + B) - sup(A)$ is an upper bound of B.

Therefore $sup(B) \leq sup(A + B) - sup(A)$ by definition of sup.

Therefore $sup(A + B) \geq sup(A) + sup(B)$ (2)

(1)and(2) jointly implies $sup(A + B) = sup(A) + sup(B)$.

QED [hb]

(b)

part b is very similar to part a.

(≥)

$$\begin{aligned}
& a \geq \inf A, b \geq \inf B, \forall a \in A, b \in B \\
& \implies a + b \geq \inf A + \inf B \\
& \implies (\inf A + \inf B) \text{ is a lower bound of } A + B \\
& \implies \inf(A + B) \geq \inf A + \inf B
\end{aligned}$$

(≤)

$$\begin{aligned}
& a + b \geq \inf(A + B) \ \ \forall a \in A, b \in B \\
& \implies a \geq \inf(A + B) - b \\
& \implies \inf(A + B) - b \text{ is a lower bound of } A, \forall b \in B \\
& \implies \inf A \geq \inf(A + B) - b \\
& \implies b \geq \inf(A + B) - \inf A \\
& \implies \inf(A + B) - \inf A \text{ is a lower bound of } B \\
& \implies \inf B \geq \inf(A + B) - \inf A \\
& \implies \inf A + \inf B \geq \inf(A + B)
\end{aligned}$$

Since $\inf A + \inf B \geq \inf(A+B)$ and $\inf A + \inf B \leq \inf(A+B)$, it must be that $\inf A + \inf B = \inf(A+B)$. QED

7.5

(a)

$$\begin{aligned}
\sqrt{n^2+1} - n &= (\sqrt{n^2+1} - n)\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} \\
&= \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} \\
&= \frac{1}{\sqrt{n^2+1}+n} \\
&= \frac{1/n}{1+\sqrt{1+\frac{1}{n^2}}}
\end{aligned}$$

In the limit of big n, the above expression obviously tends to 0.

(b)

$$\begin{aligned}
\sqrt{n^2+n} - n &= (\sqrt{n^2+n} - n)\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} \\
&= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \\
&= \frac{n}{\sqrt{n^2+n}+n} \\
&= \frac{1}{1+\sqrt{1+\frac{1}{n}}}
\end{aligned}$$

In the limit of big n, the above expression obviously tends to $\frac{1}{2}$

(c)

$$\begin{aligned}
\sqrt{4n^2+n} - 2n &= (\sqrt{4n^2+n} - 2n)\frac{\sqrt{4n^2+n}+2n}{\sqrt{4n^2+n}+2n} \\
&= \frac{4n^2+n-4n^2}{\sqrt{4n^2+n}+2n} \\
&= \frac{n}{\sqrt{4n^2+n}+2n} \\
&= \frac{1}{2+\sqrt{4+\frac{1}{n}}}
\end{aligned}$$

In the limit of big n, the above expression obviously tends to $\frac{1}{2+\sqrt{4}} = \frac{1}{4}$