HW10

33.4 let $f(x) = \begin{cases} 1, & x \text{ rational} \\ -1, & x \text{ irrational} \end{cases}$

WTS f is not integrable

let $P = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be partition of $[0,1]$

let Upper Darboux sum be
$$U(f,P) = \sum_{k=1}^{n} \sup\{f(x) \mid x \in [t_{k-1}, t_k]\} \cdot (t_k - t_{k-1})$$
$$= \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$
$$= \sum_{k=1}^{n} 1 \cdot (t_k - t_{k-1})$$
$$= 1$$

let Lower Darboux sum be
$$L(f,P) = \sum_{k=1}^{n} \inf\{f(x) \mid x \in [t_{k-1}, t_k]\} \cdot (t_k - t_{k-1})$$
$$= \sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$
$$= \sum_{k=1}^{n} (-1) \cdot (t_k - t_{k-1})$$
$$= -1$$

So we have $U(f) = \inf\{U(f,P) \mid P \text{ is a partition of } [0,1]\} = 1$

$L(f) = \sup\{L(f,P) \mid P \text{ is a partition of } [0,1]\} = -1$

Since $U(f) = 1 \neq L(f) = -1$. So f is not integrable.

WTS: $|f|$ is integrable

$$U(|f|, P) = \sum_{k=1}^{n} \sup\{|f|(x) \mid x \in [t_{k-1}, t_k]\} \cdot (t_k - t_{k-1})$$
$$= \sum_{k=1}^{n} M(|f|, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$
$$= \sum_{k=1}^{n} 1 \cdot (t_k - t_{k-1})$$
$$= 1$$

$$L(|f|, P) = \sum_{k=1}^{n} \inf\{|f|(x) \mid x \in [t_{k-1}, t_k]\} \cdot (t_k - t_{k-1})$$
$$= \sum_{k=1}^{n} m(|f|, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$
$$= \sum_{k=1}^{n} 1 \cdot (t_k - t_{k-1})$$
$$= 1$$

So we have $U(|f|) = \inf\{U(|f|, P) \mid P \text{ is a partition of } [0,1]\} = 1$

$L(|f|) = \sup\{L(|f|, P) \mid P \text{ is a partition of } [0,1]\} = 1$

Since $U(|f|) = L(|f|) = 1$. So $|f|$ is integrable.

33.7 a) Show $U(f^2, P) - L(f^2, P) \leq 2B[U(f,P) - L(f,P)]$

let f be a bounded function on $[a,b]$, $\exists B > 0$ st $|f(x)| \leq B$ $\forall x \in [a,b]$.

For $\forall x, y \in [a,b]$, we have

$$\begin{aligned} f(x)^2 - f(y)^2 &= (f(x)+f(y)) \cdot (f(x)-f(y)) \\ &\leq (|f(x)| + |f(y)|) \cdot (f(x)-f(y)) \\ &\leq (B+B) \cdot (f(x)-f(y)) \\ &\leq 2B \cdot (f(x)-f(y)) \end{aligned}$$

for $S = (t_k, t_{k+1}) \subseteq [a,b]$,

$$\begin{aligned} \sup_{x\in S}\{f(x)^2 - f(y)^2\} &\leq \sup\{2B \cdot (f(x)-f(y))\} \\ \sup_{x\in S}\{f(x)^2\} - f(y)^2 &\leq 2B \cdot \sup\{|f(x)|\} - 2B f(y) \\ M(f^2, S) - f(y)^2 &\leq 2B \cdot M(f,S) - 2B f(y) \\ \sup_{y\in S}\{M(f^2,S) - f(y)^2\} &\leq \sup_{y\in S}\{2B \cdot M(f,S) - 2B f(y)\} \\ \{M(f^2,S) - \inf_{y\in S}(f(y))^2\} &\leq \{2B \cdot M(f,S) - 2B \inf_{y\in S} f(y)\} \\ M(f^2,S) - m(f^2,S) &\leq 2B\{M(f,S) - m(f,S)\} \\ U(f^2,P) - L(f^2,P) &\leq 2B\{U(f,P) - L(f,P)\} \end{aligned}$$

b) Show if f is integrable on $[a,b]$, then f^2 also integrable on $[a,b]$.

If f is integrable, then for each $\varepsilon > 0$, $\exists \delta > 0$ st. $\text{mesh}(P) < \delta \Rightarrow$ $U(f,P) - L(f,P) < \frac{\varepsilon}{2B}$, $\forall P \in [a,b]$. From a), we have $U(f^2,P) - L(f^2,P) \leq 2B\{U(f,P) - L(f,P)\} = 2B \cdot (\frac{\varepsilon}{2B}) = \varepsilon$.

So, $\forall \varepsilon > 0$, $\exists \delta > 0$ st. $\text{mesh}(P) < \varepsilon$ implies $U(f^2,P) - L(f^2,P) < \varepsilon$, for $\forall P \in [a,b]$. hence, f^2 is integrable on $[a,b]$.

33.13 let $f, g \in [a, b]$ be continuous functions st. $\int_a^b f = \int_a^b g$.

Then $f-g$ are also continuous.

By IVT, $f-g$ is continuous on $[a, b]$, then $\exists x \in (a, b)$ st

$$(f-g)(x) = \frac{1}{b-a}\int_a^b (f-g)(t)\, dt$$

$$f(x)-g(x) = \frac{1}{b-a}\int_a^b f(t)-g(t)\, dt$$

$$= \frac{1}{b-a}\int_a^b f(t)\, dt - \int_a^b g(t)\, dt$$

$$= \frac{1}{b-a}\int_a^b f(t)\, dt - \int_a^b f(t)\, dt \quad \because \int_a^b f = \int_a^b g$$

$$= \frac{1}{b-a} \cdot 0$$

$$= 0$$

$\because f(x)-g(x) = 0$

$\therefore f(x) = g(x)$

35.4 a) $F(t) = \sin(t)$ differentiable ✓, continuous ✓

$F'(t) = \cos(t)$ continous ✓

By Thm 35.29,

$$\int_a^b f\, dF = \int_a^b f \cdot F'(x)\, dx$$

$$\int_0^{\frac{\pi}{2}} x\, dF = \int_0^{\frac{\pi}{2}} x \cdot \cos x\, dx$$

$$= \left[x\sin x + \cos x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) - \cos(0)$$

$$= \frac{\pi}{2}(1) + 0 - 1$$

$$= \frac{\pi}{2} - 1$$

u	dv
x	$+ \cos x$
1	$- \sin x$

$x\sin x - \int \sin x\, dx$

$x\sin x + \cos x \Big|_0^{\frac{\pi}{2}}$

b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\, dF = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x\, dx$

$$= \left[x\sin x + \cos x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\cdot\sin\left(-\frac{\pi}{2}\right)\right) - \cos\left(-\frac{\pi}{2}\right)\Big)$$

$$= \frac{\pi}{2} + 0 + \frac{\pi}{2}(-1)$$

$$= \frac{\pi}{2} - \frac{\pi}{2}$$

$$= 0$$

35.9 a) Let f is continuous on $[a, b]$. Then f is bounded on $[a, b]$ and $\exists c, d \in [a, b]$ st $f(c) \leq f(x) \leq f(d)$, $\forall x \in [a, b]$.

Then

$$\int_a^b f(c)\, dF \leq \int_a^b f\, dF \leq \int_a^b f(d)\, dF$$

$$f(c)\int_a^b dF \leq \int_a^b f\, dF \leq f(d)\int_a^b dF$$

$$f(c)\left[F(b) - F(a)\right] \leq \int_a^b f\, dF \leq f(d)\left[F(b) - F(a)\right]$$

$$f(c) \leq \frac{\int_a^b f\, dF}{F(b) - F(a)} \leq f(d)$$

By IVT, $\exists x \in [a, b]$ st. $f(x) = \frac{1}{F(b) - F(a)} \int_a^b f\, dF$.

Hence, $\int_a^b f\, dF = f(x)\cdot\left[F(b) - F(a)\right]$.