HW11

34.2 a) $\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\,dt$

let $F(x) = \int_0^x e^{t^2}\,dt$.

Then by Fundamental Thm of Calc. II,

$$\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\,dt = \lim_{x\to 0}\frac{F(x)}{x} = F'(0) = e^{x^2}\Big|_{x=0} = 1.$$

b) $\lim_{h\to 0} \frac{1}{h}\int_3^{3+h} e^{t^2}\,dt$

let $F(x) = \int_3^x e^{t^2}\,dt$ so that $F(3) = 0$ and $F'(x) = e^{x^2}$.

Then $\lim_{h\to 0}\frac{1}{h}\int_3^{3+h} e^{t^2}\,dt = \lim_{h\to 0}\frac{F(3+h)-F(3)}{h} = F'(3) = e^{3^2} = e^9$.

34.5 let f be a continuous function on $\mathbb{R}$.

let $g(x) = \int_1^x f(t)\,dt$ is differentiable and $g'(x) = f(x)$.

Then $F(x) = \int_{x-1}^{x+1} f(t)\,dt$

$$= \int_{x-1}^{1} f(t)\,dt + \int_1^{x+1} f(t)\,dt$$

$$= \int_1^{x+1} f(t)\,dt - \int_1^{x-1} f(t)\,dt$$

$$= \underbrace{\underset{\text{differentiable}}{g(x+1)} - \underset{\text{differentiable}}{g(x-1)}}$$

$\therefore$ F is a differentiable

$F'(x) = g'(x+1) - g'(x-1) = f(x+1) - f(x-1)$

34.7 $\int_0^1 x\cdot\sqrt{1-x^2}\,dx = ?$ use change of variables

$u = 1-x^2$

$\frac{du}{dx} = -2x\,dx \Rightarrow x\,dx = -\frac{1}{2}du$, $\int_{u=1}^{u=0}$

$$\int_0^1 x\cdot\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_1^0 \sqrt{u}\,du$$
$$= -\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^0$$
$$= \frac{1}{3}$$

Rudin 16 $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$

a) Prove $\zeta(s) = s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx$

$$s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx = s\sum_{n=1}^{\infty} n\int_n^{n+1}\frac{1}{x^{s+1}}\,dx$$
$$= s\sum_{n=1}^{\infty} n\cdot\int_n^{n+1} x^{-s-1}\,dx$$
$$= \sum_{n=1}^{\infty} n\cdot x^{-s}\Big|_n^{n+1}$$
$$= \sum_{n=1}^{\infty} n\cdot\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)$$
$$= 1\left(\frac{1}{1^s}-\frac{1}{2^s}\right)+2\left(\frac{1}{2^s}-\frac{1}{3^s}\right)+3\left(\frac{1}{3^s}-\frac{1}{4^s}\right)+\cdots$$
$$= \frac{1}{1^s}-\frac{1}{2^s}+\frac{2}{2^s}-\frac{2}{3^s}+\frac{3}{3^s}-\frac{3}{4^s}+\frac{4}{4^s}+\cdots$$
$$= \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots$$
$$= \sum_{n=1}^{\infty}\frac{1}{n^s}$$
$$= \zeta(s)$$

b) $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{x-[x]}{x^{s+1}}\,dx$

Prove integral in (b) converges for $\forall s>0$

$$\frac{s}{s-1} - s\int_1^\infty \frac{x-[x]}{x^{s+1}}\,dx = \frac{s}{s-1} - s\int_1^\infty \frac{x}{x^{s+1}}\,dx - s\int_1^\infty -\frac{[x]}{x^{s+1}}\,dx \qquad \text{from a)}$$

$$= \frac{s}{s-1} - s\left(\int_1^\infty \frac{1}{x^s}\,dx\right) + s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx$$

$$= \frac{s}{s-1} - s\cdot\left(\frac{1}{1-s}\cdot\frac{1}{x^{s-1}}\Big|_1^\infty\right) + s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx$$

$$= \frac{s}{s-1} - s\cdot\left(0-\frac{1}{1-s}\cdot\frac{1}{1^{s-1}}\right) + s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx$$

$$= \frac{s}{s-1} - s\cdot\left(\frac{1}{s-1}\right) + s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx$$

$$= s\cdot\int_1^\infty \frac{[x]}{x^{s+1}}\,dx$$

$$= \zeta(s)$$

let $f:[0,1]\to\mathbb{R}$. $f(x)=\begin{cases}0 & \text{if } x=0\\ \sin(\frac{1}{x}) & \text{if } x\in(0,1]\end{cases}$

let $\alpha:[0,1]\to\mathbb{R}$. $\alpha(x)=\begin{cases}0 & \text{if } x=0\\ \sum\limits_{n\in\mathbb{N},\ \frac{1}{n}<x} 2^{-n} & \text{if } x\in(0,1]\end{cases}$

<u>WTS</u> f is integrable wrt α on $[0,1]$.

Since $f(x)$ is bounded on $[0,1]$. From the given $f(x)$ piecewise function, which has a discontinuity at $x=0$. And because $\lim\limits_{n\to\infty}\sum\limits_{n>\frac{1}{n}} 2^{-n} = \frac{1}{2^\infty} \to 0$, so α is continuous at every point at which f is discontinuous. Hence, by Thm 6.10, $f\in R(\alpha)$, i.e. f is integrable.