HW1

Aaron Zaks
Math 104
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Ross: 1.10, 1.12
read 'rational zero thing', Ross 2.1, 2.2, 2.7

(1.10)

1.10 Prove $(2n+1)+(2n+3)+(2n+5)+\cdots+(4n-1)=3n^2$ for all positive integers n.

By MMI.

Base Cases

$n=1$: $3 = 3 = 3(1)^2$ ✓

$n=2$: $5+7 = 12 = 3(2)^2$ ✓

$n=3$: $\underbrace{7+9+11}_{n \text{ terms}} = 27 = 3(3)^2$ ✓ $\quad = \sum_{i=n}^{2n-1} 2i+1$

$n=4$: $9+11+13+15 = 48 = 3(4)^2$ ✓

Inductive Step

Sps true for $n=k$. $\sum_{i=k}^{2k-1}(2i+1) = 3k^2$. WTS: $\sum_{i=k+1}^{2k+1}(2i+1) = 3(k+1)^2$

Inductive Proof

$$\sum_{i=k+1}^{2k+1}(2i+1) = -(2k+1) + \sum_{i=k}^{2k-1}(2i+1) + \sum_{i=2k}^{2k+1}(2i+1) = 3k^2+(4k+1)+(4k+3)-2k-1$$
$$= 3k^2+6k+3$$
$$= 3(k^2+2k+1) = 3(k+1)^2$$

splitting "sides" off

We have shown $P(k) \Rightarrow P(k+1)$.

Thus by MMI, $\forall\, n \geq 1$, $\sum_{i=n}^{2n-1} 2i+1 = 3n^2$.

1.12 For $n \in \mathbb{N}$, let $n!$ [read "n factorial"] denote the product $1 \cdot 2 \cdot 3 \cdots n$. Also let $0! = 1$ and define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for} \quad k = 0, 1, \ldots, n. \tag{1.1}$$

The *binomial theorem* asserts that

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$
$$= a^n + na^{n-1}b + \frac{1}{2}n(n-1)a^{n-2}b^2 + \cdots + nab^{n-1} + b^n.$$

(a) Verify the binomial theorem for $n = 1$, 2, and 3.

(b) Show $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for $k = 1, 2, \ldots, n$.

(c) Prove the binomial theorem using mathematical induction and part (b).

a) $\underline{n=1}$

$\binom{1}{0} = 1 \quad \binom{1}{1} = 1$

$(a+b)^1 = a+b = \binom{1}{0}a^1b^0 + \binom{1}{1}a^0b^1 \quad \checkmark$

$\underline{n=2}$

$\binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1$

$(a+b)^2 = a^2 + 2ab + b^2 = \binom{2}{0}a^2b^0 + \binom{2}{1}a^1b^1 + \binom{2}{2}a^0b^2 \quad \checkmark$

$\underline{n=3}$

$\binom{3}{0} = 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = \binom{3}{0}a^3b^0 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}a^0b^3 \quad \checkmark$

b) Pf: $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

$$\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!(n-k+1)}{k!(n-k+1)!} + \frac{n! \cdot k}{k!(n-k+1)!} = \frac{n!(n-\cancel{k}+1+\cancel{k})}{k!(n+1-k)!}$$

$$= \frac{n!(n+1)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}$$

$k!((n+1)-k)! \quad k!((n+1)-k)!$

QED $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

c) Base Cases: see (a)

Inductive Step

Sps true for $n=m$. $\sum_{i=0}^{m} \binom{m}{i} a^{m-i} b^{i} = (a+b)^{m}$.

$$(a+b)^{m+1} = (a+b)(a+b)^{m} = (a+b)\sum_{i=0}^{m}\binom{m}{i}a^{m-i}b^{i}$$

$$= \sum_{i=0}^{m}\binom{m}{i}a^{(m+1)-i}b^{i} + \underbrace{\sum_{i=0}^{m}\binom{m}{i}a^{m-i}b^{i+1}}_{\text{phase shift: } i\to i-1}$$

$$= \sum_{i=0}^{m}\binom{m}{i}a^{(m+1)-i}b^{i} + \sum_{i=1}^{m+1}\binom{m}{i-1}a^{m+1-i}b^{i}$$

$$= \binom{m}{0}a^{m+1} + \sum_{i=1}^{m}\binom{m}{i}a^{(m+1)-i}b^{i} + \sum_{i=1}^{m}\binom{m}{i-1}a^{m+1-i}b^{i} + \binom{m}{m}b^{m+1}$$

$$= 1\cdot a^{m+1} + \sum_{i=1}^{m}\Big(\underbrace{\binom{m}{i}+\binom{m}{i-1}}_{\text{use pt (b)}}\Big)a^{m+1-i}b^{i} + 1\cdot b^{m+1}$$

$$= a^{m+1} + \sum_{i=1}^{m}\binom{m+1}{i}a^{(m+1)-i}b^{i} + b^{m+1} = \underline{\sum_{i=0}^{m+1}\binom{m+1}{i}a^{(m+1)-i}b^{i}}$$

re-integrate into sum

QED by MMI: $(a+b)^{n} = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^{i}$

Rational Root Thm: for $a_n x^n + \cdots + a_0 = 0$, where $a_i \in \mathbb{Z}$,

all roots $\in \mathbb{Q}$ are of the form: $\frac{p}{q}$, where $q | a_n$ and $p | a_0$.

ex) for $3x^2 + 2x - 6$. $p = \pm 1, \pm 2, \pm 3, \pm 6$ $q = \pm 1, \pm 3$ $x = \pm 1, \pm 2, \pm 3, \pm 6, \pm 1/3, \pm 2/3$

2.1 Show $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{24}$, and $\sqrt{31}$ are not rational numbers.

2.2 Show $\sqrt[3]{2}$, $\sqrt[7]{5}$ and $\sqrt[4]{13}$ are not rational numbers.

2.1 $\sqrt{3}$ is a sol of $x^2 - 3 = 0$. By RRT, all possible rational roots are $\pm 1, \pm 3$.

x	$x^2 - 3$
1	-2
-1	-2
3	6
-3	6

none are sols. Thus, $\sqrt{3}$ cannot be rational, as $x^2 - 3 = 0$ has NO rational sol.

2.7 Show the following irrational-looking expressions are actually rational numbers: (a) $\sqrt{4+2\sqrt{3}}-\sqrt{3}$, and (b) $\sqrt{6+4\sqrt{2}}-\sqrt{2}$.

2.7 a) let $a=\sqrt{4+2\sqrt3}-\sqrt3$

$$a^2=\left(\sqrt{4+2\sqrt3}-\sqrt3\right)^2=4+2\sqrt3-2\sqrt3\sqrt{4+2\sqrt3}+3$$

$$a^2=7+2\sqrt3-2\sqrt3\left(\sqrt{4+2\sqrt3}\right)$$

$$\Rightarrow a^2-7=2\sqrt3\left(1-\sqrt{4+2\sqrt3}\right)$$

$$\Rightarrow (a^2-7)^2=12\left(1-2\sqrt{4+2\sqrt3}+4+2\sqrt3\right)=60+24\overbrace{\left(\sqrt3-\sqrt{4+2\sqrt3}\right)}^{-a}$$

$$\Rightarrow (a^2-7)^2=60-24a$$

$$\Rightarrow a^4-14a^2+49-60+24a=0$$

$$\Rightarrow a^4-14a^2+24a-11=0$$

By RRT, $a=\pm1,\pm11$, so $a\in\mathbb{Q}$.

b) let $b=\sqrt{6+4\sqrt2}-\sqrt2$

$$\Rightarrow b^2=6+4\sqrt2-2\sqrt2\sqrt{6+4\sqrt2}+2=8+2\sqrt2\left(2-\sqrt{6+4\sqrt2}\right)$$

$$\Rightarrow b^2-8=2\sqrt2\left(2-\sqrt{6+4\sqrt2}\right)$$

$$\Rightarrow (b^2-8)^2=8\left(4-4\sqrt{6+4\sqrt2}+6+4\sqrt2\right)=80-32\overbrace{\left(\sqrt{6+4\sqrt2}-\sqrt2\right)}^{b}$$

$$\Rightarrow (b^2-8)^2=80-32b$$

$$\Rightarrow b^4-16b^2+64-80-32b=b^4-16b^2-32b-16=0$$

By RRT: $b=\pm1,\pm2,\pm4,\pm8,\pm16$, so $b\in\mathbb{Q}$

Ross 3.6, 4.11, 4.14, 7.5

3.6

a) Pf: $|a+b+c| \le |a|+|b|+|c|$

Recall Δ-inequality: $|a+b| \le |a|+|b|$

Addition is associative: $a+b+c = (a+b)+c = a+(b+c)$

$$|a+b+c| = |(a+b)+c| \underbrace{\le}_{\Delta\text{-inequality}} |a+b|+|c| \underbrace{\le}_{\Delta\text{-inequality}} |a|+|b|+|c|$$

QED: $|a+b+c| \le |a|+|b|+|c|$

b) Pf by MMI: $\left|\sum_{i=1}^{n} a_i\right| \le \sum_{i=1}^{n} |a_i|$

Base cases:

$n=1$: $|a_1| = |a_1|$, so $|a_1| \le |a_1|$ ✓

$n=2$: Δ-inequality, proven using linear algebra

$n=3$: proven above

Inductive Step

Sps this prop. holds for up to k numbers. That is, $\left|\sum_{i=1}^{k} a_i\right| = \sum_{i=1}^{k} |a_i|$

Inductive Proof: Show $P(k) \to P(k+1)$

$$|a_1+\dots+a_k+a_{k+1}| = |(a_1+\dots+a_k)+a_{k+1}| \underbrace{\le}_{\Delta\text{-inequality}} |a_1+\dots+a_k| + |a_{k+1}| \underbrace{\le}_{\text{inductive hypothesis}} |a_1|+\dots+|a_k|+|a_{k+1}|$$

So it holds for k+1, given it is true for k.

$\boxed{\text{By MMI, we conclude: } \forall n \in \mathbb{Z}^+, \ \left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|}$

4.11 $a, b \in \mathbb{R}$, $a < b$. Use Denseness of $\mathbb{Q}$ to show there are inf. many rationals btw. a and b.

By contradiction: suppose S is the set of rational numbers $\in (a,b)$, and that it is finite. All non-empty finite sets have a minimum. Let $\min S = r_1$. Because $\mathbb{Q} \subset \mathbb{R}$, $r_1 \in \mathbb{R}$. By the Density of $\mathbb{Q}$, there must be another rational num. btw a and r_1. Therefore, r_1 is not $\min(S)$. This is a contradiction, so we conclude S cannot be a finite set. Thus, there are inf. many rationals btw. a and b.

4.14 $A, B \subset \mathbb{R}$ nonempty, bounded $\quad A+B = \{a+b \mid a \in A, b \in B\}$

a) $\forall a \in A, \ a \leq \sup(A)$
$\forall b \in B, \ b \leq \sup(B)$

$\therefore \underline{a+b \leq \sup(A)+\sup(B)}$ Thus $\sup(A)+\sup(B)$ is an upperbound of $A+B$. We must show it is the least upper bound.

take some $\varepsilon > 0$, $\exists a, b \in A, B$ s.t. $a \geq \sup(A) - \frac{\varepsilon}{2}$ $\quad b \geq \sup(B) - \frac{\varepsilon}{2}$

$\therefore \exists a+b \in A+B$ s.t. $a+b \geq \sup(A)+\sup(B) - \varepsilon$. $\sup(A)+\sup(B)$ must be least upper bound.

4.14 b) $\inf(S) = -\sup(-S)$ where $-S = \{-s \in S\}$ in textbook

by (a): $\inf(A+B) = -\sup(-(A+B)) = -\sup(-A+(-B)) = -\sup(-A) - \sup(-B) = \inf(A) + \inf(B)$

7.5 a) $\lim_{n\to\infty} \sqrt{n^2+1} - n = \lim_{n\to\infty} \frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n} = \frac{\cancel{n^2}+1-\cancel{n^2}}{\sqrt{n^2+1}+n} = \frac{1}{+\infty} = \boxed{0}$

b) $\lim_{n\to\infty} \sqrt{n^2+n} - n = \lim_{n\to\infty} \frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n} = \frac{\cancel{n^2}+n-\cancel{n^2}}{\sqrt{n^2(1+1/n)}+n} = \frac{\cancel{n}}{\cancel{n}(\sqrt{1+1/n}+1)} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}+1}$

$= \frac{1}{\sqrt{1}+1} = \boxed{1/2}$

c) $\lim_{n\to\infty} \sqrt{4n^2+n} - 2n = \lim_{n\to\infty} \frac{(\sqrt{4n^2+n}-2n)(\sqrt{4n^2+n}+2n)}{\sqrt{4n^2+n}+2n} = \lim_{n\to\infty} \frac{\cancel{4n^2}+n-\cancel{4n^2}}{\sqrt{4n^2+n}+2n} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(\sqrt{4+1/n}+2)}$

$= \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$