HW2

9.9 Suppose there exists N_0 st. $S_n \leq t_n$ $\forall n > N_0$.

a) Prove if $\lim S_n = +\infty$, then $\lim t_n = +\infty$

let $\lim S_n = +\infty$.
then $M > 0$, there exist a number N_1 st. $n > N_1$ implies
$S_n > M$.
let $N = \max\{N_0, N_1\}$. then $n > N$
$t_n \geqslant S_n > M$

$\therefore \lim t_n = +\infty$. ☐

b) Prove if $\lim t_n = -\infty$, then $\lim S_n = -\infty$.

let $\lim t_n = -\infty$.
then $M < 0$, $\exists N_1$ st. $n > N_1$ implies
$t_n < M$.

let $N = \min\{N_0, N_1\}$. then $n > N$
$S_n \leq t_n < M$

$\therefore \lim S_n = -\infty$. ☐

c) Prove if $\lim S_n$ and $\lim t_n$ exists, then $\lim S_n \leq \lim t_n$.

WTS: $\lim S_n \leq \lim t_n$

- S_n is not finite
  - (i) $\lim S_n = -\infty$ $\lim S_n \leq \lim t_n$ always hold $a \in t_n$, $-\infty \leq a \leq \infty$
  - (ii) $\lim S_n = +\infty$ part a) answers.

- t_n is not finite
  - (i) $\lim t_n = -\infty$ part b) answers.
  - (ii) $\lim t_n = +\infty$ $\lim S_n \leq \lim t_n$ $a \in S_n$, $-\infty \leq a \leq \infty$

• finite $\lim S_n$ and t_n

Let $s = \lim S_n$, $t = \lim t_n$.

Suppose $s, t \in \mathbb{R}$ and $\lim t_n = t$.

let $\varepsilon > 0$.

Then there exist $N_1, N_2 \in \mathbb{N}$ st.

$|S_1 - s| < \varepsilon \quad \forall n > N_1$; $\quad |S_2 - t| < \varepsilon \quad \forall n > N_2$

$-\varepsilon < S_1 - s < \varepsilon \qquad -\varepsilon < S_2 - t < \varepsilon$

$-\varepsilon + s < S_1 < \varepsilon + s \qquad -\varepsilon + t < S_2 < \varepsilon + t$

let $N = \max\{N_0, N_1, N_2\}$. then

$s - \varepsilon < S_n \leq t_n < t + \varepsilon \qquad \forall n > N$.

$s < S_n \leq t_n < t + 2\varepsilon$

$\therefore \; s < t + 2\varepsilon$

Hence $\lim S_n \leq \lim t_n$.

□

9.15 how $\lim_{n \to \infty} \frac{a^n}{n!} = 0 \quad \forall a \in \mathbb{R}$.

let $S_n = \frac{a^n}{n!}$

$$\lim_{n \to \infty} \left| \frac{S_{n+1}}{S_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{a^{n+1}}{(n+1)!}}{\left(\frac{a^n}{n!}\right)} \right|$$

$$= \lim_{n \to \infty} \left| \frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{a}{n+1} \right|$$

$$= \infty \qquad \forall a \in \mathbb{R}$$

$$< 1$$

$\therefore \lim S_n = 0$.

□

10.7 S bounded nonempty subset of $\mathbb{R}$ st. sup S is not in S.

Prove there is a sequence (s_n) of points in S st. $\lim s_n = \sup S$.

sup S = upper bound of S

$\because \frac{1}{n} > 0$

$\sup S - \frac{1}{n} \neq$ upper bound

$\therefore \exists\ s_n$ st.

$\sup S - \frac{1}{n} \leq s_n \leq \sup S$

$\sup S - \frac{1}{n} \leq s_n \leq \sup S + \frac{1}{n}$

$\lim_{n\to\infty} \sup S - \frac{1}{n} = \sup S - 0 = \sup S$

$\lim_{n\to\infty} \sup S + \frac{1}{n} = \sup S + 0 = \sup S$

by squeeze thm,

hence $\lim s_n = \sup S$. $\square$

10.8 let (s_n) be increasing sequence of positive numbers and $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$.

Prove (σ_n) is an increasing sequence.

$s_{n+1} > s_n\ \forall n$

WTS: $\sigma_{n+1} - \sigma_n > 0$

$$\sigma_{n+1} - \sigma_n = \frac{1}{n+1}(s_1 + s_2 + \cdots + s_{n+1}) - \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$$

$$= \frac{n(s_1 + s_2 + \cdots + s_{n+1}) - (n+1)(s_1 + s_2 + \cdots + s_n)}{n(n+1)}$$

$$= \frac{n\cancel{(s_1+s_2+\cdots+s_n)} + ns_{n+1} - \cancel{n(s_1+s_2+\cdots+s_n)} - (s_1 + s_2 + \cdots + s_n)}{n(n+1)}$$

$$= \frac{1}{n(n+1)} \cdot \left(ns_{n+1} - (s_1 + s_2 + \cdots + s_n)\right)$$

$\because s_n$ is increasing seq $\Rightarrow s_{n+1} \geq s_n$

$\therefore ns_{n+1} - s_n \geq 0 \quad \forall n.$

Hence $\sigma_{n+1} - \sigma_n \geq 0$. σ_n is inc. seq. $\square$

10.9 let $s_1 = 1$ and $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2$ $n \geq 1$.

a) Find s_2, s_3 and s_4.

$$s_2 = s_{1+1} = \left(\frac{1}{1+1}\right) s_1^2 = \left(\frac{1}{2}\right)(1)^2 = \frac{1}{2}$$

$$s_3 = s_{2+1} = \left(\frac{2}{3}\right) s_2^2 = \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)^2 = \frac{1}{6}$$

$$s_4 = s_{3+1} = \left(\frac{3}{4}\right) s_3^2 = \left(\frac{3}{4}\right)\left(\frac{1}{6}\right)^2 = \frac{1}{48}$$

b) Show $\lim s_n$ exists.

Sequence s_n is decreasing from a). That is, $s_{n+1} < s_n$.

Prove by induction.

Basis: $n=1$ $\quad s_2 < s_1$ true
$\frac{1}{2} < 1$

Induction: assume $s_{k+1} < s_k$ is true $\forall n$. Show $s_{k+2} < s_{k+1}$ is true.

$$\begin{aligned} s_{k+2} &= s_{(k+1)+1} \\ &= \frac{k+1}{(k+1)+1} s_{k+1}^2 \\ &= \frac{k+1}{k+2} s_{k+1}^2 \\ &< s_{k+1} \end{aligned}$$

$\therefore s_{n+1} < s_n$ $\forall n$.

$\because s_1 = 1 > 0$, and if $s_n > 0$, then $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2 > 0$.

$\therefore$ decreasing and bounded sequence.

Hence this sequence converges by monotone convergence thm.

c) Prove $\lim s_n = 0$.

s_n is decreasing and $\lim s_n < s_1 = 1$. (part b) let $\lim S_n = S$.

then $\lim S_{n+1} = S$ converge to same limit

$$= \lim \frac{n}{n+1} S_n^2$$

$$= \lim \frac{n}{n+1} \left(\lim S_n^2\right)$$

$$= 1 \cdot S^2$$

$$= S^2$$

$$S = S^2$$

$$S^2 - S = 0$$

$$S = 0 \text{ or } 1$$

$\because \lim S_n < 1 \quad \neq 1$

$\therefore \lim S_n = 0$.

10.10 let $S_1 = 1$, $S_{n+1} = \frac{1}{3}(S_n + 1)$ $n \geq 1$.

a) Find S_2, S_3, S_4.

$$S_2 = S_{1+1} = \frac{1}{3}(S_1 + 1) = \frac{1}{3}(2) = \frac{2}{3}$$

$$S_3 = S_{2+1} = \frac{1}{3}(S_2 + 1) = \frac{1}{3}\left(\frac{2}{3} + 1\right) = \frac{5}{9}$$

$$S_4 = S_{3+1} = \frac{1}{3}(S_3 + 1) = \frac{1}{3}\left(\frac{5}{9} + 1\right) = \frac{14}{27}$$

b) Use induction to show $S_n > \frac{1}{2}$ $\forall n$.

P_n: $S_n > \frac{1}{2}$.

basis: $n = 1$ $S_1 = 1 > \frac{1}{2}$ true.

induction: assume $S_n > \frac{1}{2}$ is true $\forall n$. Show $S_{n+1} > \frac{1}{2}$ is true.

$$S_{n+1} = \frac{1}{3}(S_n + 1)$$

$$> \frac{1}{3}\left(\frac{1}{2} + 1\right) = \frac{1}{2}$$

$\therefore$ $S_{n+1} > \frac{1}{2}$ is true

Hence $S_n > \frac{1}{2}$ true $\forall n$. $\square$

c) show (S_n) is a decreasing sequence.

$\because S_n > \frac{1}{2} \quad \forall n.$ by b)

$2S_n > 1$

$3S_n > S_n + 1$

$S_n > \frac{1}{3}(S_n+1)$

$S_n > S_{n+1} \quad \forall n.$

$\therefore (S_n)$ is a dec. seq.

$\square$

d) show $\lim S_n$ exists and find $\lim S_n$.

$\because S_n > \frac{1}{2} \quad \forall n$ by b)

$\therefore (S_n)$ is bounded below.

$\because (S_n)$ is decreasing $\rightarrow$ monotone by c)

$\therefore$ by Monotone Convergence thm, (S_n) converges

$\therefore \lim S_n$ exists.

Find $\lim S_n$:

let $\lim S_n = s$.

$= \lim S_{n+1}$

$s = \frac{1}{3}s + \frac{1}{3}$

$\frac{2}{3}s = \frac{1}{3}$

$s = \frac{1}{2}$

$\therefore \lim S_n = \frac{1}{2}.$

10.11 let $t_1 = 1$, $t_{n+1} = \left(1 - \frac{1}{4n^2}\right) \cdot t_n$ $\quad n \geq 1$.

a) Show $\lim t_n$ exists.

Observe $0 < t_n \leq 1$ $\forall n$. $\quad \because 1 - \frac{1}{4n^2}$

$\uparrow$

$0 < \# < 1$

$\therefore$ t_n is bounded. $\quad \underbrace{\qquad}_{0 < t_n < 1}$

WTS t_n is decreasing.

$$t_{n+1} - t_n = \left(1 - \frac{1}{4n^2}\right) \cdot t_n - t_n$$
$$= t_n\left(1 - \frac{1}{4n^2} - 1\right)$$
$$= t_n\left(-\frac{1}{4n^2}\right) \qquad \because + \cdot - = -$$
$$< 0$$

$\therefore$ $t_{n+1} < t_n$

$\therefore$ t_n is decreasing.

$\therefore$ t_n converges. (Thm 10.2)

Hence $\lim t_n$ exists.

b) $\lim t_n = ?$

t_n is bounded decreasing seq. part a)

Induction : $P(n)$ $0 < t_n \leq 1$

Basis : $n=1$ $\quad t_1 = 1 > 0$ true.

Induction : assume $0 < t_n$ true $\forall n \in \mathbb{N}$. then

$4n^2 > 1 \quad \frac{1}{4n^2} < 1 \quad 1 - \frac{1}{4n^2} > 0$ $\qquad 1 - \frac{1}{4n^2}$

$\uparrow$

$0 < \# < 1$

$t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n > 0$ $\qquad \underbrace{\qquad}_{0 < t_n < 1}$

$\therefore$ t_{n+1} true.

$\therefore$ $0 < \lim t_n \leq 1$

Squeeze thm proof

Let a_n, b_n, C_n be 3 sequences, st. $a_n \leq b_n \leq C_n$, and $L = \lim a_n = \lim C_n$.

Show that $\lim b_n = L$.

WTS: if for every number $\varepsilon > 0$, there is some number $\delta > 0$ st.
$|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$.

$\lim_{x \to c} a_n = L$.

Any $\varepsilon > 0$, $\delta_1 > 0$ when $0 < |x-c| < \delta_1$,

then $|a_n - L| < \varepsilon$.

$\lim C_n = L$.

Any $\varepsilon > 0$, $\delta_2 > 0$ when $0 < |x-c| < \delta_2$,

then $|C_n - L| < \varepsilon$.

when $0 < |x-c| < \delta_3$,

$a_n \leq b_n \leq C_n$.

let $\delta = \min(\delta_1, \delta_2, \delta_3)$ ← making a smaller interval

min

For any $\varepsilon > 0$, there exists $\delta > 0$ st.

If $0 < |x-c| < \delta$, then

$|a_n - L| < \varepsilon$

$L - \varepsilon < a_n < L + \varepsilon$

$|C_n - L| < \varepsilon$

$L - \varepsilon < C_n < L + \varepsilon$

$a_n \leq b_n \leq C_n$

$\Rightarrow L - \varepsilon < b_n < L + \varepsilon$

$|b_n - L| < \varepsilon$

$\therefore \lim_{x \to c} b_n = L$.

$\square$