HW3 10.6, 11.2, 11.3, 11.5

10.6

a) $|S_{n+1} - S_n| < 2^{-n} \quad \forall n \in \mathbb{N}$

Prove S_n is Cauchy seq and hence a convergent seq.

let $a_n = \sum_{n}^{\infty} \frac{1}{2^n}$.

By Geometric series, $a_n = \dfrac{\frac{1}{2^n}}{1-\frac{1}{2}} = \dfrac{\frac{1}{2^n}}{\frac{1}{2}} = \frac{1}{2^n} \cdot 2 = \frac{\cancel{2}\,1}{2^n \cdot 2^{-1}} = \frac{1}{2^{n-1}}$

$\lim_{n\to\infty} a_n \to 0$

By def, Cauchy seq $\Rightarrow$ $\varepsilon > 0$, $\exists N$ st. $|S_n - S_m| < \epsilon \quad m, n > N$.

let $m > n$, $\varepsilon > 0$. (w/o loss of generality)

$$|S_m - S_n| = |S_m - S_{m-1} + S_{m-1} - S_{m-2} + \cdots - S_{n+1} + S_{n+1} - S_n|$$

$$\leq |S_m - S_{m-1}| + |S_{m-1} - S_{m-2}| + \cdots + |S_{n+1} - S_n| \qquad \text{(by triangle inequality)}$$

$$\leq 2^{-m+1} + 2^{-m+2} + \cdots + 2^{-n}$$

$$< a_n$$

$$< \epsilon$$

$\therefore (S_n)$ is a Cauchy seq.

By Lemma 10.10, all cauchy seq are bounded. Hence convergent.

b) Is a) result true if we only assume $|S_{n+1} - S_n| < \frac{1}{n} \quad \forall n \in \mathbb{N}$?

No. $$|S_{n+1} - S_n| = \left|1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n} + \tfrac{1}{n+1} - \left(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n}\right)\right|$$

$$= \left|\cancel{1} + \cancel{\tfrac{1}{2}} + \cdots + \cancel{\tfrac{1}{n}} + \tfrac{1}{n+1} - \cancel{1} - \cancel{\tfrac{1}{2}} - \cdots - \cancel{\tfrac{1}{n}}\right|$$

$$= \frac{1}{n+1}$$

$$< \frac{1}{n}$$

$S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is <u>not convergent</u> $\forall n \in \mathbb{N}$.

$\therefore S_n$ is not Cauchy seq.

11.2 $a_n = (-1)^n$ $b_n = \frac{1}{n}$ dec. $c_n = n^2$ inc. $d_n = \frac{6n+4}{7n-3}$ dec. $\overset{d_{n+1}}{>} \frac{6(n+1)+4}{7(n+1)-3} = \frac{6n+10}{7n+4}$

a) example of monotone subseq = ?

$a_{n_k} = (-1)^{2k} = 1$

$b_{n_k} = \frac{1}{n}$

$c_{n_k} = n^2$

$d_{n_k} = \frac{6n+4}{7n-3}$

b) set of subsequential limit = ?

$a_n : S = \{-1, +1\}$

$b_n : S = \{0\}$

$c_n : S = \{+\infty\}$

$d_n : S = \{\frac{6}{7}\}$

c) limsup = ? liminf = ?

$\limsup a_n = \sup S = 1$

$\liminf a_n = \inf S = -1$

$\limsup b_n = \liminf b_n = 0$

$\limsup c_n = \liminf c_n = +\infty$

$\limsup d_n = \liminf d_n = \frac{6}{7}$

b, c, d only contain 1 elem.

d) converges ? diverges to $+\infty$, $-\infty$?

$b \to 0$ a do not conv.

$d \to \frac{6}{7}$ c div. to $+\infty$

e) bounded ?

a, b, d bounded c unbounded

11.3 $\quad S_n = \cos\left(\frac{n\pi}{3}\right) \qquad t_n = \frac{3}{4n+1} \qquad U_n = \left(-\frac{1}{2}\right)^n \qquad V_n = (-1)^n + \frac{1}{n}$

t_{n+1}

$> \frac{3}{4(n+1)+1}$

dec.

$\Omega = \{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \cdots\}$

$\Omega = \{0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, -\frac{4}{5}, \cdots\}$

$n=1 \quad (-1)^1 + \frac{1}{1} = -1+1 = 0$

$n=2 \quad (-1)^2 + \frac{1}{2} = 1+\frac{1}{2} = \frac{3}{2}$

$n=3 \quad (-1)^3 + \frac{1}{3} = -1+\frac{1}{3} = \frac{-2}{3}$

$n=4 \quad (-1)^4 + \frac{1}{4} = \frac{5}{4}$

$n=5 \quad (-1)^5 + \frac{1}{5} = -\frac{4}{5}$

a) example of monotone subseq = ?

$\Omega = \{1, \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \cdots\}$

$a_{n_k} = \cos\left(\frac{6n\pi}{3}\right) = \cos(2n\pi) = \{1, 1, 1, \cdots\}$

$t_{n_k} = \frac{3}{4n+1}$

$U_{n_k} = \left(-\frac{1}{2}\right)^{2n} = \{\frac{1}{4}, \frac{1}{16}, \cdots\}$

$V_{n_k} = (-1)^{2n} + \frac{1}{2n} = \{\frac{3}{2}, \frac{5}{4}, \cdots\}$

b) set of subsequential limit = ?

$S_n : S = \{-1, -\frac{1}{2}, \frac{1}{2}, 1\} \qquad t_n : S = \{0\} \qquad U_n : S = \{0\} \qquad V_n : S = \{-1, 1\}$

c) limsup = ? $\quad$ liminf = ?

$\limsup S_n = 1$

$\liminf S_n = -1$

$\limsup t_n = \liminf t_n = 0$

$\limsup U_n = \liminf U_n = 0$

$\limsup V_n = 1$

$\liminf V_n = -1$

d) converges ? $\quad$ diverges to $+\infty$, $-\infty$?

t_n, U_n converge. $\qquad$ S_n, V_n do not converge.

e) bounded ?

all bounded.

11.5 let (q_n) be an enumeration of all rationals in the interval $(0,1]$.

a) set of subsequential limits $= ?$

Let (s_n) be a sequence.

(i) If t is in $\mathbb{R}$, then there is a subsequence of (s_n) converging to t if and only if the set $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ is infinite for all $\epsilon > 0$.

by Thm 11.2, suppose $t \in [0,1]$. let $\epsilon > 0$.

by Denseness of $\mathbb{Q} \in \mathbb{R}$, # of rationals in interval $(a-\epsilon, a+\epsilon) = \infty$

$\Rightarrow$ set $\{n : |q_n - t| < \epsilon\}$ has ∞ terms.

$\therefore$ all elems of set $[0,1]$ are subsequential limits.

0 is also a subsequential limit since interval $(0, 0+\epsilon)$ contains ∞ elems of q_n $\forall \epsilon > 0$.

b) $\limsup q_n = ?$ $\liminf q_n = ?$

$\limsup q_n = 1$ $\qquad$ $\liminf q_n = 0$.

Q: What is limsup ?

What is the difference btw limsup and sup ?

What is most counter-intuitive abt limsup ?

Can you state some sentences that seems to be correct, but is actually wrong?

limsup is the tail of the sup ; the largest possible limit after a large number N.
Whereas, sup is the least upper bound, sup can also view as the max.
The thing abt limsup that's most counter-intuitive is that sup always greater or equal to limsup. Sometimes even though a sequence S_n does not always converge, limsup always exists or equals ∞.

ex: $S_n = \{7, 2, 8, 5, 6, \cdots, 1, 0, 1, 0, 1, 0, \cdots\}$

this seq doesn't converge, but we have sup = 8 and limsup = 1.