HW6

1. <u>WTS</u> $[0,1]^2 \in \mathbb{R}^2$ is Sequentially compact.

<u>Proof</u> Let $(a_n, b_n) \in X \times Y$ be given where $X \subset M$ and $Y \subset N$ are sequentially compact. There exists a subsequence (a_{n_k}) that converges to some point $a \in X$ as $k \to \infty$. The subsequence (b_{n_k}) has a sub-subsequence $(b_{n_{k(\ell)}})$ that converges to some $b \in Y$ as $\ell \to \infty$.

The sub-subsequence $(a_{n_{k(\ell)}})$ continues to converge to the point a.

Thus, $(a_{n_{k(\ell)}}, b_{n_{k(\ell)}}) \to (a, b)$ as $\ell \to \infty$.

This implies that $X \times Y$ is sequentially Compact. $\square$

Hence, $[0,1]^2 \in \mathbb{R}^2 = ([0,1], [0,1]) \in X \times Y$, is sequentially compact.

2. WTS E is not countable.

let $X = 0.a_1a_2 \cdots a_n \cdots$ be the decimal expansion of $X \in E$ where a_i can only be 4 or 7 let assume E is countable. Then there exist a bijection in E. let construct a new decimal expansion by using Cantor diagonalization argument On the decimal expansion of X, call this new decimal expansion b, whose b_i entries is constructed from taking the opposite # from the one on X's diagonal.

For example,
$$\begin{aligned} X_1 &= 0.d_{11}d_{12} \cdots d_{1n} \cdots = 0.7747 \cdots \qquad b = 0.4774 \cdots \\ X_2 &= 0.d_{21}d_{22} \cdots d_{2n} \cdots = 0.7477 \cdots \\ &\vdots \qquad\qquad \ddots \qquad\qquad = 0.4744 \cdots \\ &\qquad\qquad\qquad\qquad\quad = 0.4447 \cdots \end{aligned}$$

In this way, b is different from every element $X_i \in E$. That is, $b \notin E$. Implies it is not bijective in E. Hence E is not countable. $\square$

WTS E is compact.

Since E can only consist of 4 and 7, hence the max will be 0.777... (upper bound) and the min will be 0.444... (lower bound). Hence E is bounded.

Since the complement of E is the union of open intervals, and so if E^c is open, then E is closed.

Hence E is compact by the Heine-Borel thm. $\square$

3. WTS there is some elems live in $\overline{B}$, but not $\bigcup_i \overline{A_i}$.

let A_i be a subset of $\mathbb{R} = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. then $\overline{A_i} = \{\frac{1}{n}\} = A_i$.

assume $B = \bigcup_i A_i = \bigcup_i \overline{A_i} = \bigcup_n^\infty \frac{1}{n} \neq 0$ implies $0 \notin B$.

$\overline{B} = \bigcup_n^\infty \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$ implies $0 \in \overline{B}$.

Hence $\overline{B} \supset \bigcup_i \overline{A_i}$ is a strict inclusion.

4. The argument is not true b/c a closed subset of $\mathbb{R}$ can also be an uncountable union of closed intervals, recall what we see in problem #2, set E is closed but uncountable, which can also serve as an counterexample of this claim.