HW7

1. If X and Y are open cover compact, then $X \times Y$ is open cover compact.

Since X and Y are open cover compact.

Then X and Y have finite subcover. that is,

Let $X = \{X_k\}_{k=1}^{n}$ be finite, and $\{U_i\}$ be an open cover of X.

$Y = \{Y_t\}_{t=1}^{n}$ be finite, and $\{Q_i\}$ be an open cover of Y.

Pick $U_1 \in \{U_i\}$ to contain X_1. $Q_1 \in \{Q_i\}$ to obtain Y_1.

Pick $U_2 \in \{U_i\}$ to contain X_2. $Q_2 \in \{Q_i\}$ ... Y_2

Pick $U_k \in \{U_i\}$ to contain X_k ... etc. $Q_t \in \{Q_i\}$ ... Y_t

Obtain $\{U_k\}_{k=1}^{n} \subseteq \{U_i\}$ $\{Q_t\}_{t=1}^{n} \subseteq \{Q_i\}$

$\therefore$ cover X, finite cover Y, finite

$X \times Y = \{X_k\}_{k=1}^{n} \times \{Y_t\}_{t=1}^{n}$

$= \{U_k\}_{k=1}^{n} \times \{Q_t\}_{t=1}^{n}$ cover X and Y, still finite

Hence $X \times Y$ is open cover compact.

2. $f: X \to Y$. continuous map btw metric space. $A \subset X$ subset.

T/F.

- if A is open, then f(A) is open.

$\boxed{\text{False}}$ ① $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 5$. Every open set is mapped to $\{5\}$, not open.

- if A is closed, then f(A) is closed.

$\boxed{\text{False}}$ ② $f(x) = \frac{1}{1+x^2}$, $X = \mathbb{R}$, $A = [0, \infty)$ is closed,

but $(0, 1]$ is not closed.

· if A is bounded, then $f(A)$ is bounded.

$\boxed{\text{False}}$ $f(x) = \frac{1}{x}$, $X = \mathbb{R} \setminus \{0\}$, $A =$ bounded set $(0,1) \subset X$.

but $f(A) = [1, \infty)$ is not bounded.

· if A is compact, then $f(A)$ is compact.

$\boxed{\text{True}}$ WTS: $\exists$ a seq. (y_n) in $f(A)$ has a subseq. converges to $f(A)$.

let (y_n) be a seq in $f(A)$. Then $\exists (x_n) \in A$ st. $(y_n) = f(x_n)$.

Since A is compact. Then (x_n) has a convergent subseq $(x_{n_k}) \to x$.

Since $(x_{n_k}) \to x$ and f is continuous, then $(y_{n_k}) = f(x_{n_k}) \to f(x)$.

Since $x \in A \subset X$, then $f(x) \in f(A)$, and (y_n) has a convergent subseq $(y_{n_k}) \to f(x)$.

Hence $f(A)$ is compact.

· if A is connected, then $f(A)$ is connected.

$\boxed{\text{True}}$ suppose not, A is connected, but $f(A)$ is disconnected, $f(A) = B \cup C$.

$B, C \neq \phi$ nonempty, open, $B \cap C = \phi$.

let $B' = f^{-1}(B)$, $C' = f^{-1}(C)$. open, nonempty.

$$B' \cap C' = f^{-1}(B) \cap f^{-1}(C)$$
$$= f^{-1}(B \cap C) = f^{-1}(\phi) = \phi$$

$\therefore B' \cup C' = f^{-1}(B) \cup f^{-1}(C) = f^{-1}(B \cup C) = f^{-1}(f(A)) = A$.

so $A = B'$ and C' are seperated contradicts A is connected. $\square$

3. Prove there is no continuous map $f:[0,1]\to\mathbb{R}$ st. f is surjective.
(there is a surjective map from $(0,1)\to\mathbb{R}$ though).

Suppose not, such function $f:[0,1]\to\mathbb{R}$ exists.
Since $[0,1]$ is compact and $f:[0,1]\to\mathbb{R}$ is continous, image $f([0,1])$ is compact. since f is surjective, $f([0,1])=\mathbb{R}$, hence $\mathbb{R}$ is compact, contradiction.