HW 8

1. $f_n(x) = \dfrac{n+\sin x}{2n+\cos n^2 x}$, show f_n converges uniformly on $\mathbb{R}$.

$$\lim_{n\to\infty} \left| \frac{1+\frac{1}{n}\sin x}{2+\frac{1}{n}\cos n^2 x} \right| = \left| \frac{1+0}{2+0} \right| = \frac{1}{2}$$

$\therefore f_n(x) \to \frac{1}{2}$ pointwise on $\mathbb{R}$.

WTS: $f_n(x) \to \frac{1}{2}$ uniformly on $\mathbb{R}$.

$$\left| f_n(x) - \frac{1}{2} \right| = \left| \frac{n+\sin x}{2n+\cos n^2 x} - \frac{1}{2} \right| = \left| \frac{2\sin x + \cos n^2 x}{2(2n+\cos n^2 x)} \right| \underset{\text{by triangular inequality}}{\leq} \frac{3}{2(2n-1)}$$

$\sup\{ |f_n(x) - \frac{1}{2}| : x \in \mathbb{R} \} \leq \frac{3}{2(2n-1)}$, $n \in \mathbb{N}$

Since $\lim_{n\to\infty} \left| \frac{3}{2(2n-1)} \right| \to 0$, hence we get $f_n \to \frac{1}{2}$ uniformly on $\mathbb{R}$.

2. $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Show series is continuous on $[-1,1]$ if $\sum_n |a_n| < \infty$.
Prove $\sum_{n=1}^{\infty} n^{-2} x^n$ is continuous on $[-1,1]$.

WTS $\sum_{n=1}^{\infty} a_n x^n$ is cont. on $[-1,1]$.
Since $|a_n x^n| \leq |a_n|$ for $x \in [-1,1]$ and $\sum_n |a_n| < \infty$,
by Weierstrass M-test, $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[-1,1]$.
Since $a_n x^n$ is continuous for each n, by Rudin Thm 7.12,
limit of $\sum a_n x^n$ is continuous on $[-1,1]$.

WTS $\sum_{n=1}^{\infty} n^{-2} x^n$ is cont. on $[-1,1]$.
Since $|n^{-2} x^n| \leq |n^{-2}|$ for $[-1,1]$ and $\sum_n |\frac{1}{n^2}| < \infty$,
by Weierstrass M-test, $\sum_{n=1}^{\infty} |n^{-2} x^n|$ converges uniformly on $[-1,1]$.

Since $n^{-2}x^n$ is continuous for $\forall n$, then by Thm 7.12,
lim of $\sum n^{-2}x^n$ is continuous on $[-1, 1]$.

3. Show $f(x) = \sum_n x^n$ represent a continuous function on $(-1, 1)$, but the convergence is not uniform.

(Hint: show $f(x)$ on $(-1, 1)$ is continuous, you only need to show that for any $0 < a < 1$, we have uniform convergence on $[-a, a]$. use Weierstrass M-test.)

$$f(x) = \sum_n x^n$$

$$d_\infty(f, 0) = \sup_{x \in (-1,1)} |f(x) - 0| = \sup_x |x^n - 0| = 0.9\bar{9} \approx 1 \not\to 0$$

$\therefore$ the convergence is not uniform.